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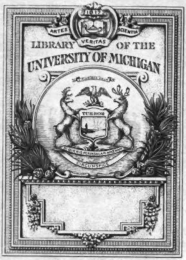

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TREATISE

ON

MENSURATION,

BOTH IN

THEORY and PRACTICE.

By CHARLES HUTTON.

NEWCASTLE UPON TYNE:

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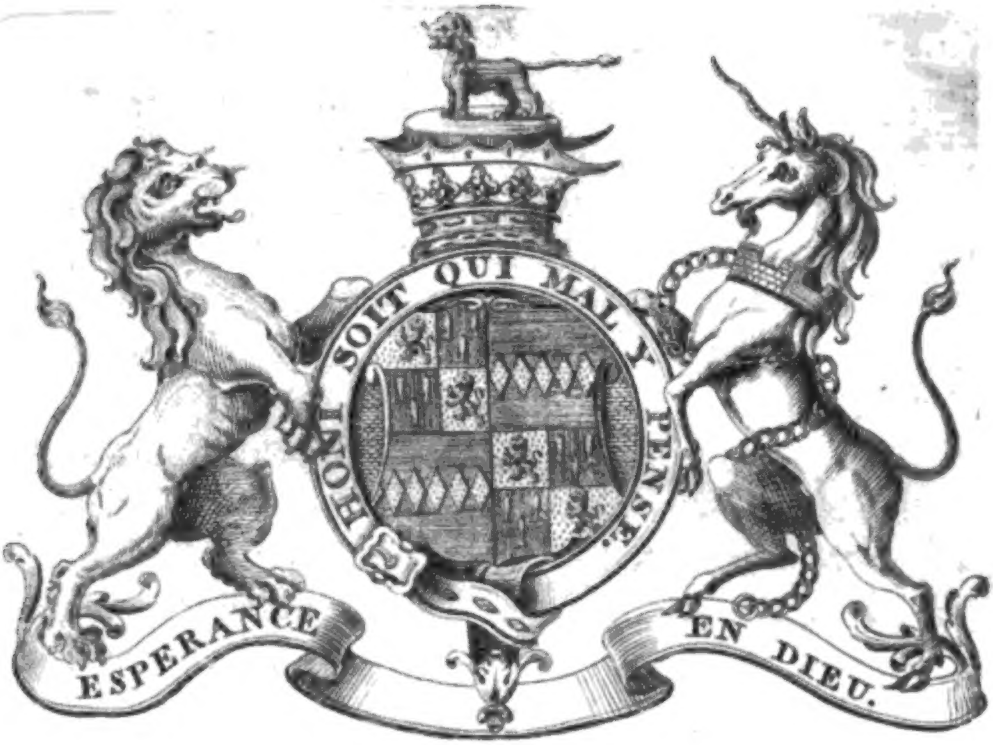
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TO
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MY LORD,

I AM well aware how preſumptive the world will think me in being deſirous of having the name of one of Your illuſtrious rank, and univerſally confeſſed abilities in every branch of literature, prefixed to the performance which I now have the honour to preſent to Your Grace: Yet I hope it will do me juſtice, and not think that I am weak enough, in ſo doing, to have the vanity to ſuppoſe I had any

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any thing to offer in this work unknown to Your Grace. But if the work have any merit in itſelf, and may be of any uſe to thoſe leſs converſant in the liberal arts and ſciences than Your Grace, my intention, I hope, will fully apologize for my ambition. For Your name, I am convinced, will attract the attention of many, who might not otherwiſe look into the work; and ſhould ſuch receive any benefit or advantage from the peruſal of it, I am ſure, Your Grace will feel that ſatisfaction which none but the *real* Patrons of the ſcientific arts, and who have the *real* Intereſt and Love of their Country *truly* at heart, can feel.

To acknowledge Obligations which it is an Honour to have received, would ſtill carry with it the appearance of Greater Ambition; and an attempt to deſcribe thoſe Virtues and Perfections which Your Grace's Public Works and Public Services have rendered ſo conſpicuous, would be a Weakneſs indeed!

Permit me, then, humbly to aſſure Your Grace, that I have and ſhall ever retain the moſt grateful ſenſe of the Favours and Honour done me; and that I am,

With the profoundeſt Reſpect,
YOUR GRACE'S
Moſt obedient and
Moſt devoted humble Servant,
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Mr T. Todd, Weſt Smithfield, London
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— Tweddale
— Thibou
— Tomlinſon, Land Surveyor, Aſton, Warwickſhire
— John Thomſon, Schoolmaſter, Berwick
— Rob. Thompſon, Haſwel Moor
— P. Turner, Hayes, Middleſex
— W. Taylor, Schoolmaſter, Southall, Middleſex
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— William Tate, Schoolmaſter, Togſton
— Turton, Walker
— John Taylor, Schoolmaſter, Newcaſtle
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— Joſeph Turner, Blyth Shades
— William Taylor, Dodworth
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— Auguſtus Tillet, Norwich
— Thompſon, Schoolmaſter, Hull
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V

The Hon. Raby Vane
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— Cha. Vyſe, Math. M. Portland Street, Cavendiſh Square

W

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Y

Mr Thomas Yelloley, Newcaſtle
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— John Yarrow, Schoolmaſter
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PREFACE.

BY Menſuration I underſtand the art and ſcience which is concerned about the meaſure of extention, or the magnitude of figures; and it is, next to arithmetic, a ſubject of the greateſt uſe and importance, both in affairs that are abſolutely neceſſary in human life, and in every branch of the mathematics: A ſubject by which ſciences are eſtabliſhed, and commerce is conducted; by whoſe aid we manage our buſineſs, and inform ourſelves of the wonderful operations in nature; by which we meaſure the heavens and the earth, eſtimate the capacities of all veſſels and bulks of all bodies, gauge our liquors, build edifices, meaſure our lands and the works of artificers, buy and ſell an infinite variety of things neceſſary in life, and are ſupplied with the means of making the calculations which are neceſſary for the conſtruction of almoſt all machines.

IT is evident that the cloſe connection of this ſubject with the affairs of men would very early evince its importance to them; and accordingly the greateſt among them have paid the utmoſt attention to it; and the chief and moſt eſſential diſcoveries in geometry in all ages, have been made in conſequence of their efforts in this ſubject. Socrates thought that the prime uſe of geometry was to meaſure the ground, and indeed this buſineſs gave name to the ſubject; and moſt of the ancients ſeem to have had no other end beſides menſuration in view in all their laboured geometrical diſquiſitions. Euclid's elements are almoſt entirely devoted to it; and although there be contained in them many properties of geometrical figures which may be applied to other purpoſes, and indeed of which the moderns have made the moſt material uſes in various diſquiſitions of exceedingly different kinds, I ſay notwithſtanding this, yet Euclid himſelf ſeems to have adapted them entirely to this purpoſe. For, if it be conſidered that his elements contain a continued chain of reaſoning and of truths, of which the former are ſucceſſively applied to the diſcovery of the latter, one propoſition depending on another, and the ſucceeding propoſitions ſtill approximating towards ſome particular object near the end of each book; and when at the laſt we find that object to be the equality, proportion, or relation between the magnitudes of figures both plane and ſolid; it is ſcarcely poſſible to avoid allowing this to have been Euclid's grand object. And accordingly he determined the chief properties in the menſuration of rectilineal plane and ſolid figures; and ſquared all ſuch planes, and cubed all ſuch ſolids. The only curve figures which he attempted beſides are the circle and ſphere; and when he could not accurately determine their meaſures, he gave an excellent method of approximating to them, by ſhewing how in a circle to inſcribe a regular polygon which ſhould not touch another circle, concentric with the former, although their circumferences ſhould be ever ſo near together; and, in like manner, between any two concentric ſpheres to deſcribe a polyhedron which ſhould not any where touch the inner one: And approximations to their meaſures are all that have hitherto been given. But although he could not ſquare the circle, nor cube the ſphere, he determined the proportion of one circle to another, and of one ſphere to another, as well as the proportions of all rectilineal ſimilar figures to one another.

Archimedes took up menſuration where Euclid left it, and carried it a great length. He was the firſt who ſquared a curvilineal ſpace, unleſs Hypocrates muſt be excepted on account of his lunes. In his time the conic ſections were admitted in geometry, and he applied himſelf cloſely to the meaſuring of them as well as other figures. Accordingly he determined the relations of ſpheres, ſpheroids, and conoids, to cylinders and cones; and the relations of parabolas to rectilineal planes whoſe quadratures had long before been determined by Euclid.—He hath left us alſo his attempts upon the circle: He proved that a circle is equal to a right-angled triangle whoſe baſe is equal to the circumference and its altitude equal to the radius; and conſequently that its area is found by drawing the radius into half the circumference; and ſo reduced the quadrature of the circle to the determination of the ratio of the diameter to the circumference; but which however hath not yet been done. Being diſappointed of the exact quadrature of the circle, for want of the rectification of its circumference, which all his methods would not effect, he proceeded to aſſign an uſeful approximation to it; this he effected by the numerical calculation of the perimeters of the inſcribed and circumſcribed polygons; from which calculations it appears, that the perimeter of the circumſcribed regular polygon of 192 ſides is to the diameter, in a leſs ratio than that of $3\frac{1}{7}$ $(3\frac{10}{70})$ to 1, and that the inſcribed polygon of 96 ſides is to the diameter in a greater ratio than that of $3\frac{10}{71}$ to 1; and conſequently much more that the circumference of the circle is to the diameter in a leſs ratio than that of $3\frac{1}{7}$ to 1, but greater than that of $3\frac{10}{71}$ to 1: The firſt ratio of $3\frac{1}{7}$ to 1, reduced to whole numbers, gives that of 22 to 7, for $3\frac{1}{7} : 1 :: 22 : 7$, which therefore will be nearly the ratio of the circumference to the diameter. From this ratio of the circumference to the diameter he computed the approximate *area* of the circle, and found it to be to the ſquare of the diameter as 11 is to 14.—He likewiſe determined the relation between the circle and ellipſe, with that of their ſimilar parts.—The hyperbola too in all probability he attempted, but it is not to be hoped that he met with any ſucceſs, ſince approximations to its area are all that can be given by all the methods that have ſince been invented.

Beſides theſe figures, he hath left us a treatiſe on the ſpiral deſcribed by a point moving uniformly along a right line, which at the ſame time moves with an uniform angular motion; and determined the proportion of its area to that of its circumſcribed circle, as alſo the proportion of their ſectors.

Throughout the whole works of this great man, which are chiefly on menſuration, he every where diſcovers the deepeſt deſign and the fineſt invention; and ſeems to have been (with Euclid) exceedingly careful of admitting into his demonſtrations nothing but principles perfectly geometrical and unexceptionable: And although his moſt general method of demonſtrating the relations of curved figures to ſtreight ones, be by inſcribing polygons in them; yet to determine thoſe relations, he does not increaſe the number and diminiſh the magnitude of the ſides of the polygon *in infinitum*; but from this plain fundamental principle, allowed in Euclid's elements, viz. that any quantity may be ſo often multiplied, or added to itſelf, as that the reſult ſhall exceed any propoſed finite quantity of the ſame kind, he proves that to deny his figures to have the propoſed relations, would involve an abſurdity.

He demonſtrated alſo many properties, particularly in the parabola, by means of certain numerical progreſſions whoſe terms are ſimilar to the inſcribed figures; but without conſidering ſuch ſeries to be continued *in infinitum*, and then ſumming up the terms of ſuch infinite ſeries.

He had another very curious and ſingular contrivance for determining the meaſures of figures, in which he proceeds as it were mechanically by weighing them.

Several other eminent men among the ancients wrote on this ſubject, both before and after Euclid and Archimedes; but their attempts were uſually upon particular parts of it, and according to methods not eſſentially different from theirs. Among theſe are to be reckoned Thales, Anaxagoras, Pithagoras, Bryſon, Antiphon, Hypocrates of Chios, Plato, Apollonius, Philo, and Ptolomy; moſt of whom wrote of the quadrature of the circle, and thoſe after Archimedes, by his method, uſually extended the approximation to a greater degree of accuracy.

Many of the moderns alſo have proſecuted the ſame problem of the quadrature of the circle, after the ſame methods, to greater lengths; ſuch are Vieta, and Metius whoſe proportion between the diameter and circumference is that of 113 to 355, which is within about $\frac{1}{100000000}$ of the true ratio; but above all, Ludolph van Ceulen, who with an amazing degree of induſtry and patience, by the ſame methods extended the ratio to 20 places of figures, making it that of 1 to 3·14159265358979323846+.

The firſt material deviation from the principles uſed by the ancients in geometrical demonſtrations was made by Cavalerius: The ſides of their inſcribed and circumſcribed figures they always ſuppoſed of a finite and aſſignable number and length; he introduced the doctrine of indiviſibles, a method which was very general and extenſive, and which with great eaſe and expedition ſerved to meaſure and compare geometrical figures. Very little new matter however was added to geometry by this method, its facility being its chief advantage. But there was great danger in uſing it, and it ſoon led the way to infinitely ſmall elements, and infiniteſimals of endleſs orders; methods which were very uſeful in ſolving difficult problems, and in inveſtigating or demonſtrating theories that are general and extenſive; but ſometimes led their incautious followers into errors and miſtakes, which occaſioned diſputes and animoſities among them. There were now however many excellent things performed in this ſubject; not only many new things were effected concerning the old figures, but new curves were meaſured; and for many things which could not be exactly ſquared or cubed, general and infinite approximating ſeries were aſſigned, of which the laws of their continuation were manifeſt, and of ſome of which the terms were independent on each other. Mr Wallis, Mr Huygens, and Mr James Gregory performed wonders; Huygens in particular muſt be admired for his ſolid, accurate, and very maſterly works.

During the preceding ſtate of things, ſeveral men, whoſe vanity ſeemed to have overcome their regard for truth, aſſerted that they had diſcovered the quadrature of the circle, and publiſhed their

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attempts in the form of ſtrict geometrical demonſtrations, with ſuch aſſurance and ambiguity as ſtaggered and miſled many who could not ſo well judge for themſelves, and perceive the fallacy of their principles and arguments. Among thoſe were Longomontanus, and our countryman Hobbs, who obſtinately refuſed all conviction of his errors.

The uſe of infinites was however diſliked by ſeveral people, and particularly by Sir Iſaac Newton, who among his numerous and great diſcoveries hath given us that of the method of fluxions; a diſcovery of the greateſt importance both in philoſophy and mathematics; it being a method ſo general and extenſive, as to include all inveſtigations concerning magnitude, diſtance, motion, velocity, time, &c. with wonderful eaſe and brevity; a method eſtabliſhed by its great author upon true and inconteſtible principles; principles perfectly conſiſtent with thoſe of the ancients, and which were free from the imperfections and abſurdities attending ſome that had lately been introduced by the moderns: he rejected no quantities as infinitely ſmall, nor ſuppoſed any parts of curves to coincide with right lines; but propoſed it in ſuch a form as admits of a ſtrict geometrical demonſtration. Upon the introduction of this method moſt ſciences aſſumed a different appearance, and the moſt abſtruſe problems became eaſy and familiar to every one; things which before ſeemed to be inſuperable, became eaſy examples or particular caſes of theories ſtill more general and extenſive; rectifications, quadratures, cubatures, tangencies, caſes *de maximis & minimis*, and many other ſubjects became general problems, and delivered in the form of general theories which included all particular caſes; thus, in quadratures, an expreſſion would be inveſtigated which defined the areas of all poſſible curves whatever, both known and unknown, and which, by proper ſubſtitutions, brought out the area for any particular caſe, either in finite terms, or in infinite ſeries of which any term or any number of terms could be eaſily aſſigned; and the like in other things. And although no curve whoſe quadrature was unſucceſsfully attempted by the ancients, became by this method perfectly quadrable, there were aſſigned many general methods of approximating to their areas, of which in all probability the ancients had not the leaſt idea or hope, and innumerable curves were ſquared which were utterly unknown to them.

The excellency of this method revived ſome hopes of ſquaring the circle, and its quadrature was attempted with eagerneſs. The quadrature of a ſpace was now reduced to the finding of the fluent of a given fluxion, but this problem however was found to be incapable of a general ſolution in finite terms; the fluxion of every fluent was always aſſignable, but the reverſe of this problem could be effected only in particular caſes; among the exceptions, to the great grief of the geometers, was included the caſe of the circle, with regard to all the forms of fluxions attending it. Another method of obtaining the area was tried: of the quantity expreſſing the fluxion of any area, in general, could be aſſigned the fluent in the form of an infinite ſeries, which ſeries therefore defined all areas in general, and which, on ſubſtituting for particular caſes, was often found to break off and terminate, and ſo afford an area in finite terms; but here again the caſe of the circle failed, its area ſtill coming out an infinite ſeries.—All hopes of the quadrature of the circle being now at an end, the geometricians employed themſelves in diſcovering and ſelecting the beſt forms of infinite ſeries for determining its area, among which it is evident that thoſe were to be preferred which were ſimple and which would converge quickly; but it generally happened that theſe two properties were divided, the ſame ſeries very rarely including them both: the mathematicians in moſt parts of Europe were now buſy, and many ſeries were aſſigned on all hands, ſome admired for their ſimplicity, and others for their rate of convergency; thoſe which converged the quickeſt, and were at the ſame time ſimpleſt, which therefore were moſt uſeful in computing the area of the circle in numbers, were thoſe in which, beſides the radius, the tangent of ſome certain arc of the circle, was the quantity by whoſe powers the ſeries converged; and from ſome of theſe ſerieſes the area hath been computed to a great extent of figures: Mr Edmund Hally gave a remarkable one from the tangent of 30 degrees, which was rendered famous by the very induſtrious Mr Abraham Sharp, who by means of it extended the area of the circle to 72 places of figures, as may be ſeen in Sherwin's book of logarithms; but even this was afterwards outdone by Mr John Machin, who, by means hereafter deſcribed in this book, compoſed a ſeries ſo ſimple and which converged ſo quickly, that by it, in a very little time, he extended the quadrature of the circle to 100 places of figures; from which it appears, that if the diameter be 1, the circumference will be 3·1415926535, 8979323846, 2643383279, 5028841971, 6939937510, 5820974944, 5923078164, 0628620899, 8628034825, 3421170679+, and conſequently the area will be ·7853981633, 9744830961, 5660845819, 8757210492, 9234984377, 6455243736, 1480769541, 0157155224, 9657008706, 3355292669+.

Whilst I have been giving the preceding account of the progreſs of this ſubject, I have at the ſame time unawares been writing its panegyric; for, from hence it appears, that all or moſt of the material improvements or inventions in the principles or method of treating of geometry, have been made eſpecially for the improvement of this chief part of it, menſuration; which abundantly ſhews, what I at firſt undertook to declare, the dignity of this ſubject; a ſubject which, as Doctor Barrow ſays, after mentioning ſome other things, "deſerves to be more curiouſly weighed, becauſe from hence a name is impoſed upon that mother and miſtreſs of the reſt of the mathematical ſciences, which is employed about magnitudes, and which is wont to be called geometry (a word taken from ancient uſe, becauſe it was firſt applied only to meaſuring the earth and fixing the limits of poſſeſſions) though the name ſeemed very ridiculous to Plato, who ſubſtitutes in its place that more extenſive name of Metrics or Menſuration; and others after him give it the title of Pantometry, becauſe it teaches the method of meaſuring all kinds of magnitudes."

But notwithſtanding the dignity and importance of this ſubject, the books which have lately been offered to the public under the title of menſuration, have treated it in a very unworthy manner, and have brought it into much contempt. My deſign in this work therefore is to place the ſubject in a more favourable light, and in ſome meaſure to retrieve its reputation; and of which I ſhall now give ſome account.

This work is firſt divided into five principal parts, each part being cut into ſeveral ſections, each of which ſections contains ſeveral problems or propoſitions. The problems and propoſitions are all demonſtrated after the ſimpleſt and ſhorteſt methods that could be diſcovered; I having proceeded ſometimes by pure geometry, and ſometimes by algebra, the method of fluxions, the method of increments, &c. according as the one or the other ſeemed to be the beſt adapted to the buſineſs then in hand. And thus, by not confining myſelf to one mode of demonſtrating, I was enabled to make my book exceedingly more compleat than it could otherwiſe be; not only by uſing that particular method which would perform the buſineſs in the ſhorteſt or cleareſt manner, but which alſo would ſerve to extend the ſubject to the greateſt length. However I had not *always* my choice of my method of demonſtrating, the ſubject ſometimes confining to one method, and ſometimes to another; for although the method of fluxions be generally the moſt conciſe, and, in other reſpects, the moſt proper method of inveſtigating the meaſure of extenſions, there are ſome things in this buſineſs that are too ſimple for it, as well as ſome others that ariſe above its reach: Thus, the method of fluxions cannot determine the meaſure of a rectangle, that determination being below its root, by being prior to the fluxionary method of determining areas; for in this method, that determination is ſuppoſed, and its reſult aſſumed in the very notation; for, the rectangle which is conſidered as the fluxion of a ſurface, is denoted by its length drawn into its breadth: And on this account thoſe fluxioniſts who, in their chapter of quadratures, attempt to determine the meaſure of a rectangle, argue in a circle, or deduce as a concluſion what was neceſſarily ſuppoſed in the premiſes. The ſame may be ſaid with regard to priſms. And, on the other hand, the ſame method alone would not extend to the determination of the equality of a triangle and hyperbola of the ſame baſe and of an infinite length, I being there obliged to call in the aſſiſtance of the method of increments.

As extenſions are of three kinds, longitudinal, ſuperficial, and ſolid; accordingly in this work the ſcience is treated of as diſtinguiſhed by nature into theſe three principal parts; that is, ſo far as the nature of geometrical figures would conveniently admit. With regard to right lines, plane ſurfaces, and ſolids, the diſtinction is general; but it does not obtain with regard to curved lines and ſurfaces. For, by preſerving this diſtinction ſo entire, as to determine the meaſure of all kinds of lines in the firſt part, and of all kinds of ſurfaces in the ſecond; I immediately perceived that I could not ſo well adapt my book to the convenient ſtudy of the generality of readers: on which account, in the firſt part I have treated only of the meaſure of right lines; and in the ſecond part, only of the meaſure of planes which are bounded by right and circular lines, without any curve ſurfaces excepting that of the ſphere. In the third part, which treats of ſolids, I have generally placed the problems which relate to the meaſures of the lines, ſurfaces, and ſolidities of each particular figure, immediately after each other, becauſe that generally the knowledge of the one of them led to *that* of the others. The latter parts of the book are employed chiefly about the applications of the general problems to ſeveral intereſting practical ſubjects in life.

So much for the diſtribution and order of the parts in general. I ſhall now proceed to deſcribe the contents of the parts themſelves more particularly.

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The whole work conſiſts of 5 parts.

Part I. Contains the menſuration of right lines and right-lined angles, and it divided into 3 ſections.

Sect. 1. Contains ſome geometrical definitions and problems. Some of the definitions are new, and, it is preſumed, more compleat and leſs exceptionable than thoſe inſtead of which they have been ſubſtituted. The problems will prepare the learner for making the ſeveral figures which afterwards are treated of.

Sect. 2. Contains plane Trigonometry, or the meaſuring of lines and angles. All the caſes of trigonometry, both right and oblique angled are here reduced to 3 only; by which means it is eaſier to be remembered, and more clearly underſtood. Beſides theſe caſes, which perform the buſineſs in the common way by means of the ſines, &c. of angles, I have given a new and extenſive method, by which all the caſes of trigonometry are performed independent of all ſines, tangents, &c. and without any kind of tables.

Sect. 3. Contains the application of trigonometry to the determination of heights and diſtances; in which a great variety of caſes and methods concerning this curious ſubject are explained.

Part II, Treats on ſuperficial menſuration, or the menſuration of plane figures, and is divided into 2 ſections.

Sect. 1. Treats on the areas, &c. of right-lined and circular figures. Here, beſides many things that are new and curious, are given an explanation of Profeſſor Machin's famous quadrature of the circle, and the demonſtrations of ſome uſeful approximations to the meaſures of circular arcs and areas, which had been given by Mr Huygens and Sir Iſaac Newton without demonſtrations.

Sect. 2. Contains a curious and uſeful collection of queſtions concerning areas, promiſcuouſly placed, and ſolved by the rules in the former ſections.

Part III. Contains the meaſuring of ſolids, and is divided into 8 ſections.

Sect. 1. Treats on bodies that are bounded by right or by circular lines, viz. priſms, pyramids, the ſphere, and the circular ſpindle.

Sect. 2. Treats on the conic ſections in general; and, though it be ſhort, contains ſeveral things that are new and of great importance.

Sect. 3. Treats on the ellipſe and the figures generated by it, viz. ſpheroids and elliptic ſpindles.

Sect. 4. In like manner treats on parabolic lines, areas, ſurfaces, and ſolidities. And

Sect. 5. On hyperbolic lines, areas, ſurfaces, and ſolidities.

In theſe ſections the ſeveral figures and bodies are very extenſively and particularly handled, many of the rules, &c. being new (which indeed is the caſe throughout the whole book) and intereſting; and I have given throughout many pretty approximations of the values of ſeveral things which cannot be truly expreſſed otherwiſe than by an infinite ſeries, which approximations are generally new, excepting two or three formerly given by Sir I. Newton, and which I have demonſtrated.

Sect. 6. Treats on the five regular ſolids or bodies. And

Sect. 7. On ſolid rings. And then

Sect. 8. Or the laſt of this part, contains a promiſcuous collection of queſtions concerning ſolids, to exerciſe the learner in the foregoing rules.

Part IV. Contains, in 3 ſections, ſeveral ſubjects relating to menſuration in general.

Sect. 1. Contains a treatiſe on the true quadrature and cubature of curves in general. In which are contained ſome of the moſt univerſal and moſt important propoſitions that can be made in the ſubject.

Sect. 2. Contains the equidiſtant-ordinate method; or, the approximate quadrature and cubature of curves in general, by means of equidiſtant ordinates or ſections. A ſubject by which general and finite rules are diſcovered for all figures; for ſome of which they are accurately true, and in the others they are exceedingly near approximations; which are often the moſt uſeful rules that can be applied to many things in real practice.

Sect. 3. Contains, in a very conciſe but copious treatiſe, the relations between the areas and ſolidities of figures, and the centers of gravity of their generating lines and planes.

Then the

Vth and laſt Part, in 4 ſections, contains the application of the general rules to the moſt uſeful ſubjects of meaſuring that happen in life. In theſe ſubjects very material improvements are almoſt every where made, both with reſpect to the matters and the diſpoſition of them.

Sect. 1. Contains a treatiſe of land ſurveying; explaining the uſe of the inſtruments, the methods of ſurveying, of planning, of computing the contents, of reducing plans, and of diviſion of ground.

Sect. 2. Contains a very curious and compleat treatiſe on gauging. As in like manner doth

Sect. 3. On the meaſuring of artificers works; viz. Bricklayers, Maſons, Carpenters and Joiners, Slaters and Tilers, Plaſterers, Painters, Glaziers, Pavers, and Plumbers. Containing the deſcription of the carpenters rule, the ſeveral meaſures uſed by each, with the methods of taking the dimenſions, and of ſquaring and ſumming them up. The whole illuſtrated by a real caſe of a building, in which are ſhewn the methods of entering the dimenſions and contents in the pocket book, of drawing out the abſtracts, and from them drawing out the forms of the bills.

Sect. 4. Contains a curious treatiſe on timber meaſuring; in which, among ſeveral other things, is given a new rule for meaſuring round timber, which not only brings out the true content ſufficiently exact, but is full as eaſy in the operation as the common falſe one either by the pen or the ſliding rule; as alſo ſome curious rules for cutting timber to the moſt advantage.

The book then concludes with a large table of the areas of circular ſegments, extended to ten times the uſual length. Other tables are ſuppreſſed on account of the great ſize to which the book hath increaſed.

It may may be neceſſary to remark that, in this book, where a curve or a ſpace is ſaid to be non-quadrable, or it is ſaid that the value of a thing cannot be expreſſed but by an infinite ſeries, or any ſuch-like expreſſion is uſed, the meaning is that it is not geometrically quadrable, or that its area or value cannot be expreſſed in a finite number of terms by any method yet known, or by the method there uſed; and not that it is a thing any way impoſſible in itſelf. For, although a ſpace be not quadrable by the methods yet known, it does not therefore follow that its quadrature is an impoſſible thing, or that ſome method may not hereafter be diſcovered by which it may be ſquared. All the methods uſed by the geometers before Archimedes were inſufficient for the quadrature of any curve ſpace whatever; but were they therefore to infer that no curve could by any means be ſquared? What have ſince been done abundantly ſhew the imprudence and falſhood of the aſſertion. Archimedes diſcovered a method by which he ſquared the parabola; and, by the lately-diſcovered method of fluxions, we can find as many quadrable curves as we pleaſe.—But whilſt I am urging the poſſibility of the quadrature of *any* ſpace, I am not ignorant of the pretenſions of ſeveral people to prove the impoſſibility of *that* of the *circle* in particular. There are attempts to demonſtrate this impoſſibility in the Leipſic Acts, as well as in our own Philoſophical Tranſactions; but thoſe demonſtrations are far from being ſufficiently general to afford me any conviction of it. On the other hand, ſeveral conſiderations may induce us to believe the poſſibility of its quadrature: This quadrature is known to depend upon the rectification of the circumference, and that a right line may be of the exact length of the circumference, may be thus conceived; In the deſcription of a circle by a pair of compaſſes, ſuppoſing the deſcribing point or leg to move uniformly, the deſcription will be performed in a certain time and with a certain velocity, which time drawn into the velocity will produce a certain length for the circumference: Again, by conceiving the circle to roll upon a right line at the ſame time when it turns round its own center, after the manner of the motion of a wheel, the right line and circumference will be congruous and equal by ſucceſſive and continual application; But without theſe illuſtrations, the circumference is evidently of ſome certain length (for it is not infinitely great) and every finite length in nature muſt have a meaſure, it being only the fault of our methods of notation or expreſſion that we have not determined it.—It may be farther added, that a ſpace which by ſome methods hath a finite expreſſion, is by others expreſſed by an infinite ſeries, as appears in page 318 and throughout ſect. 1 part 4. Even a number is the ſame way affected, and what is a finite expreſſion in one method or ſcale, is, in another, expreſſed by an infinite ſeries; thue $\frac{1}{3} = \cdot 333$ &c. $\frac{1}{6} = \cdot 166$ &c. and theſe infinite decimals, and many others in the preſent decimal ſcale, would be finite expreſſions in a duodecimal, and innumerable other ſcales; and, again, ſome numbers that are finite in the preſent ſcale, would be infinite in them. And who can prove that even the root of 2, or of any other number whoſe root in the preſent ſcale is an infinite ſeries, may not be a terminate quantity in ſome ſcale whoſe root is a ſquare number (for ſuch an one it muſt be). It is true we have not yet found the area of the circle, nor the root of 2, of 3, &c. in finite terms, yet for each of theſe we can aſſign infinite ſeries whoſe laws of progreſſion are viſible; which is more than the ancients could do, or perhaps ever expected could be done, if they even at all thought of ſuch things. And I have no doubt but that hereafter will be diſcovered a method of ſquaring any figure whatever. Which is the chief problem in geometry.

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EXAMPLE I.

What is the area of a ſector whoſe radius is 10 and arc 20?

By the rule, $\frac{20\times10}{2}=100$ the area.

EXAMPLE II.

To find the area of the ſector whoſe radius is 9 and the chord of whoſe arc is 6.

By the laſt problem the length of the arc is 6·117063.

Therefore $\frac{6.117063\times9}{2}=27{\cdot}5267835$ the area required.

$v=$ the verſed ſine of $\frac{1}{2}a$; then, by multiplying the radius by half the arc, as found by the ſeveral rules of the laſt problem, we ſhall obtain the following values of the area of the ſector.

I. $A=\frac{1}{2}ar={\cdot}01745329\,brr.$

II. $A=r\sqrt{dv}\times:1+\frac{v}{2.3d}+\frac{3v^2}{2.4.5d^2}+\frac{3.5v^3}{2.4.6.7d^3}$ &c.

III. $A=rs\times:1+\frac{s^2}{2.3r^2}+\frac{3s^4}{2.4.5r^4}+\frac{3.5s^6}{2.4.6.7r^6}$ &c.

IV. $A=\frac{4\sqrt{ss+vv}-s}{3}r=\frac{4\sqrt{2rv}-s}{3}r=\frac{4\sqrt{ss+vv}-s}{3}\times\frac{ss+vv}{2v}$, nearly.

V. $A=rd\sqrt{\frac{3v}{3d-v}}$, nearly.

VI. $A=\frac{r}{9}\times\overline{5d\sqrt{\frac{5v}{5d-3v}}+4\sqrt{dv}}$, nearly.

It is evident that the area of the ſector might be expreſſed in ſeveral other different manners; ſuch as by the tangent, coſine, &c. of its ſemi-arc; but the forms above given are the moſt uſeful ones.

RULE II.

If it be known what proportion it bears to the whole circle, or what part it is of it; Take the like proportional part of the area of the whole circle for that of the ſector.*

Note. A ſector bears the ſame proportion to the whole circle as doth the degrees in its arc or angle to 360, or as its arc to the whole circumference.

EXAMPLE I.

What is the area of a ſemicircle, and of a quadrant, the radius being 10?

Firſt, $\cdot7854 \times 10 \times 10 = 78\cdot54 =$ the whole circle.

Then $78\cdot54 \div 2 = 39\cdot27 =$ the ſemicircle.

And $78\cdot54 \div 4 = 19\cdot635 =$ the quadrant.

EXAMPLE II.

To find the area of a ſector whoſe arc or angle contains 18 degrees, to a radius of 3 feet?

Firſt, $\cdot7854 \times 3 \times 3 = 7\cdot0686 =$ the whole circle.

Then $360 : 18 :: 7\cdot0686 : \frac{7\cdot0686 \times 18}{360} = \frac{7\cdot0686}{20} = \cdot35343$ the area of the ſector required.

EXAMPLE III.

To find the area of a ſector whoſe radius is 9 and the chord of whoſe arc is 6.

By the example to rule 1 of prob. 6, the degrees in the arc were found to be $38\cdot9424412$. But $\cdot785398 \times 9 \times 9 \times 4 =$ the area of the circle.

---

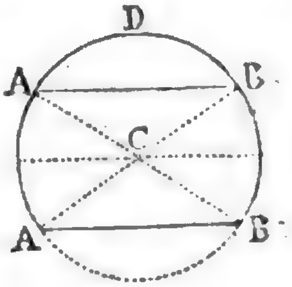
* This rule is too evident to need any formal proof.

Therefore $360 : 38{\cdot}9424412 :: {\cdot}785398 \times 9 \times 9 \times 4 : \frac{38{\cdot}9424412 \times {\cdot}785398 \times 9 \times 9 \times 4}{360} = 38{\cdot}9424412 \times {\cdot}0785398 \times 9 =$ $27{\cdot}52678388$ the area of the ſector required.

PROBLEM IX.

To find the area of the ſegment of a circle.

RULE I.



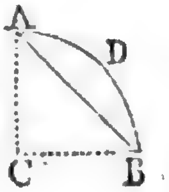
By one of the rules of the laſt problem, find the area of the ſector *ACB* having the ſame arc with the ſegment *ADB* required.

Find alſo the area of the triangle *ABC* formed by the chord of the ſegment and the two radiuſes of the ſector.

Then the ſum or difference of theſe areas will be that of the ſegment, according as it is greater or leſs than a ſemicircle.

EXAMPLE I.

What is the area of a ſegment whoſe arc is a quadrant or contains 90 degrees, the diameter being 18 feet?



Firſt, $\cdot 7854 \times 18 \times 18$ is the whole circle.

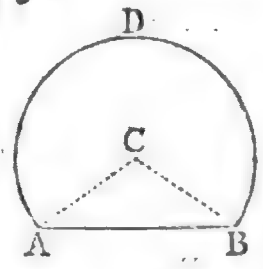
And $\frac{\cdot 7854 \times 18 \times 18}{4} = {\cdot}7854 \times 9 \times 9 =$ the quadrant or ſector *CADB*.

But $\frac{AC \times CB}{2} = \frac{9 \times 9}{2} = {\cdot}5 \times 9 \times 9 =$ the triangle *ACB*.

Therefore ${\cdot}7854 \times 9 \times 9 - {\cdot}5 \times 9 \times 9 = {\cdot}2854 \times 9 \times 9 = 23{\cdot}1174$ = the area of the ſegment required.

EXAMPLE II.

What is the area of a ſegment of a circle whoſe arc contains 280°, the diameter being 50?



Firſt, $\cdot 7854 \times 50 \times 50 = 1963{\cdot}5 =$ the whole circle.

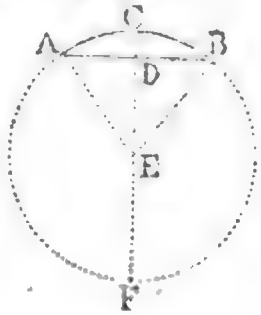
And $360 : 280 :: 1963{\cdot}5 : \frac{1963{\cdot}5 \times 280}{360} = \frac{1963{\cdot}5 \times 7}{9} = 1527{\cdot}16666 =$ the ſector *CADBC*.

Again, the angle $ACB = 360 - 280 = 80°$. And the triangle $ACB = AC \times CB \times \frac{1}{2}$ ſine $\angle ACB = 25 \times 25 \times \frac{1}{2}$ s. of $80° = 25 \times 25 \times {\cdot}4924039 = 307{\cdot}7524375$.

Wherefore the ſum $1834{\cdot}9191 =$ the ſegment *ADBA* required.

EXAMPLE III.

Required the area of the ſegment whoſe chord is 20, and height, or verſed ſine of half its arc, 2.



Since $AD^2 = DC \times DF$, therefore $DF = \frac{AD^2}{DC} = \frac{100}{2} = 50$. And the diameter $CF = FD + DC = 50 + 2 = 52$. Alſo $DE = EC - CD = \frac{1}{2} CF - CD = 26 - 2 = 24$.

Therefore the triangle $ABE = AD \times DE = 10 \times 24 = 240$.

Again, by trigonometry, $26 = AE : 10 = AD :: 1 =$ s. $\angle D : \frac{10}{26} = \frac{5}{13} = {\cdot}38461538 =$ s. $\angle AED$, or of the arc $AC = 22{\cdot}6198614$ degrees, the double of which or $45{\cdot}2397228$ are the degrees in the arc *ACB*.

But $\cdot 785398 \times 52 \times 52 =$ the whole circle.

Therefore $360 : 45{\cdot}2397228 :: {\cdot}785398 \times 52 \times 52 : \frac{45{\cdot}2397228 \times {\cdot}785398 \times 13 \times 26}{45} = 266{\cdot}8786995 =$ the ſector *AEB*.

Whence $266{\cdot}8786995 - 240 = 26{\cdot}8786995 =$ the ſegment *ACBA* required.

Otherwiſe.

Having found the triangle $= 240$, as above; we ſhall have, by rule 6 of the laſt problem, the ſector $= \overline{\frac{5\times52}{4}\sqrt{\frac{5\times2}{5\times52-3\times2}}+\sqrt{52\times2}}\times\frac{2\times52}{9} = \overline{65\sqrt{\frac{5}{127}}+2\sqrt{26}}\times\frac{104}{9}$ $= \overline{\frac{65}{127}\sqrt{635}+2\sqrt{26}}\times\frac{104}{9} = 266{\cdot}87868$.

Then the difference is $26{\cdot}87868 =$ the ſegment nearly the ſame as before.

RULE II.

Let d be the diameter, and v the verſed ſine, or the height of the ſegment;

Then $2v\sqrt{dv}\times:\frac{2}{3}-\frac{v}{5d}-\frac{v^2}{28d^2}-\frac{v^3}{72d^3}$ &c. or $2v\sqrt{dv}\times:\frac{2}{3}-\frac{3v}{5.2d}A-\frac{5v}{7.4d}B-\frac{7.3v}{9.6d}C-\frac{9.5v}{11.8d}D$ &c. $=$ the ſegment; $A, B, C,$ &c. being the firſt, ſecond, third, &c. terms.[*]

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[*] DEMONSTRATION.

By the laſt problem, the value of the ſector *CADB* (fig. in page 99) is $\frac{1}{2}d\sqrt{dv}\times:1+\frac{v}{2.3d}+\frac{3v^2}{2.4.5d^2}+\frac{3.5v^3}{2.4.6.7d^3}$ &c. But $\pm\frac{1}{2}d\mp v =$ the altitude of the triangle *ABC*, and its baſe $AB = 2\sqrt{dv-vv} = 2\sqrt{dv}\times:1-\frac{v}{2d}-\frac{v^2}{2.4d^2}-\frac{3v^3}{2.4.6d^3}$ &c. and, therefore, the area of the triangle $ABC = \overline{\pm\frac{1}{2}d\mp v}\times\sqrt{dv}\times:1-\frac{v}{2d}-\frac{v^2}{2.4d^2}-\frac{3v^3}{2.4.6d^3}$ &c. which being added to, or ſubtracted from the ſector, gives $2v\sqrt{dv}\times:\frac{2}{3}-\frac{v}{5d}-\frac{v^2}{4.7d^2}-\frac{3v^3}{4.6.9d^3}-\frac{3.5v^4}{4.6.8.11d^4}$ &c. for the value of the ſegment.

EXAMPLE.

Take, here, the laſt example, in which $d=52$ and $v=2$; then $\frac{v}{d}=\frac{1}{26}$; and the proceſs will be thus:

$$\begin{array}{llll}
+A=\frac{4v\sqrt{dv}}{3}=\frac{8}{3}\sqrt{104} & = & & +27\cdot1947707 \\
-B=\frac{3v}{5.2d}A=\frac{3}{10.26}A & = & - & \cdot3137858 \\
-C=\frac{5v}{7.4d}B=\frac{5}{7.4.26}B & = & - & \cdot0021551 \\
-D=\frac{7.3v}{9.6d}C=\frac{7}{3.6.26}C & = & - & \cdot0000322 \\
-E=\frac{9.5v}{11.8d}D=\frac{9.5}{11.8.26}D & = & - & \cdot0000007
\end{array}$$

the ſum of the negative terms $-\ \cdot3159738$ taken from the firſt term leaves $26\cdot8787969=$ the area required.

RULE

Otherwiſe.

Putting $y=$ the ſemi-chord of the ſegment, and the other letters as before. We have the fluxion of the ſegment $y\dot{v}=\dot{v}\sqrt{dv-vv}=\dot{v}\sqrt{dv}\times:1-\frac{v}{2d}-\frac{v^2}{2.4d^2}-\frac{3v^3}{2.4.6d^3}$ &c. the fluent of which gives $v\sqrt{dv}\times:\frac{2}{3}-\frac{v}{5d}-\frac{v^2}{4.7d^2}-\frac{3v^3}{4.6.9d^3}$ &c. for the value of half the ſegment, as before. *Q. E. D.*

COROLLARY I.

Let the chord of half the arc of the ſegment be denoted by c; then $d=\frac{cc}{v}$, by which exterminating d out of the above ſeries, gives $cv\times\frac{2}{3}-\frac{v^2}{5c^2}-\frac{v^4}{4.7c^4}-\frac{3v^6}{4.6.9c^6}-\frac{3.5v^8}{4.6.8.11c^8}$ &c. for the value of the ſemi-ſegment.

COROLLARY II.

Let the ſupplemental verſed ſine be denoted by V, viz. $V=d-v$; then $d=V+v$, by which exterminating d out of the ſame ſeries, will

RULE III.

If v denote the verſed ſine, as before, and V the verſed ſine of the ſupplement, or $V=d-v$; Then $\frac{4}{3}v\sqrt{vV}+\frac{5}{5V}A-\frac{1v}{7V}B+\frac{3v}{9V}C-\frac{5v}{11V}D$ &c. will be = the ſegment; where A, B, C, &c. denote the preceding terms.

EXAMPLE.

Taking, again, the ſame example; we have $v=2$, and $V=52-2=50$. Hence $\frac{v}{V}=\frac{2}{50}=\cdot04$, and

$A=$

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will produce $2v\sqrt{Vv}\times:\frac{1}{1.1.3}+\frac{v}{1.3.5V}-\frac{v^2}{3.5.7V^2}+\frac{v^3}{5.7.9V^3}-\frac{v^4}{7.9.11V^4}$ &c. for the value of the ſaid ſemi-ſegment. And this ſeries, which is rule 3, converges very quickly in ſmall ſegments.

COROLLARY III.

By ſuppoſing the ſeveral indeterminate quantities in theſe ſeries to be expreſſed by any given numbers, we may obtain an endleſs variety of infinite ſeries whoſe ſums will be given. And, in particular, if the diameter be 1, and ſuppoſing the verſed ſines to be equal to the radius or the diameter, we ſhall obtain the following ſeries, in which n denotes $\cdot785398$ &c. the area of the whole circle whoſe diameter is 1, or $\frac{1}{4}$ of the circumference of the ſame circle.

I. $n=\sqrt{2}\times:\frac{2}{3}-\frac{1}{5.2}-\frac{1}{4.7.2^2}-\frac{1.3}{4.6.9.2^3}-\frac{1.3.5}{4.6.8.11.2^4}$ &c.

II. $n=2\times:\frac{2}{3}-\frac{1}{5}-\frac{1}{4.7}-\frac{1.3}{4.6.9}-\frac{1.3.5}{4.6.8.11}$ &c.

III. $n=\frac{1}{1.1.3}+\frac{1}{1.3.5}-\frac{1}{3.5.7}+\frac{1}{5.7.9}-\frac{1}{7.9.11}$ &c.

Several other values of n may be aſſigned from the corollaries in pages 29 and 30, and from pages 85, &c.

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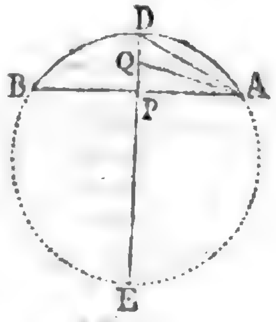
$$A=\tfrac{4}{3}v\sqrt{vV}=+\ 26{\cdot}66666666\tfrac{2}{3}$$
$$B=\frac{v}{5V}A\ =\ +\ \ 21333333\tfrac{1}{3}$$
$$C=\frac{v}{7V}B\ =\ -\ \ 121904\tfrac{16}{21}$$
$$D=\frac{3v}{9V}C\ =\ +\ \ 1625\tfrac{8}{21}$$
$$E=\frac{5v}{11V}D\ =\ -\ \ 29\tfrac{12}{21}$$
$$F=\frac{7v}{13V}E\ =\ +\ \ \tfrac{13}{21}$$

ſum of the affir. terms $+\ 26{\cdot}88001626$
ſum of the negat. terms $-\ \ 0{\cdot}00121934\tfrac{1}{3}$
the difference is $26{\cdot}87879691\tfrac{2}{3}$ = the area required.

RULE IV.

To the chord of the whole arc add $\frac{4}{3}$ of the chord of half the arc, multiply the ſum into the verſed ſine or height of the ſegment, and $\frac{4}{15}$ of the product will be the area of the ſegment nearly.



That is, $\overline{C+\frac{4}{3}c}\times\frac{4}{15}v$, or $\overline{C+\frac{4}{3}\sqrt{dv}}\times\frac{4}{15}v$, or $\overline{C+\frac{4}{3}\sqrt{\frac{1}{4}C^2+v^2}}\times\frac{4}{15}v$, or $\overline{\sqrt{dv-vv}+\frac{2}{3}\sqrt{dv}}\times\frac{4}{5}v$ = the area nearly; where $d=DE$ the diameter, $C=BA$ the chord of the whole arc, $c=AD$ the chord of the half arc, and $v=DP$ the height of the ſegment.*

Ex-

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* DEMONSTRATION.

The ſegment is $=2v\sqrt{dv}\times:\frac{2}{3}-\frac{v}{5d}-\frac{v}{28d}$ &c. by the laſt rule.

Suppoſe

EXAMPLE.

Taking, ſtill, the ſame example, in which $C=20$, and $v=2$.

Then $\overline{C+\frac{4}{3}\sqrt{\frac{1}{4}CC+vv}}\times\frac{4}{15}v=26{\cdot}877908=$ the area nearly the ſame as before.

RULE V.

Draw AQ [ſee the laſt figure] ſo that $DP:PQ::5:\sqrt{10}$, then the ſegment $ADBA$ will be nearly equal to $\frac{4}{3}DP\times AQ=\frac{4}{3}v\sqrt{dv-\frac{3}{5}vv}=\frac{4}{3}v\sqrt{cc-\frac{3}{5}vv}=\frac{4}{3}v\sqrt{\frac{1}{4}CC+\frac{2}{5}vv}$. Where $d=$ the diameter, $v=DP$, $c=DA$, and $C=AB$, as in the laſt rule.*

Suppoſe it $=2DP\times\overline{m\times PA+n\times AD}=2v\times\overline{m\sqrt{dv-vv}+n\sqrt{dv}}=2v\sqrt{dv}\times\overline{m\sqrt{1-\frac{v}{d}}+n}=2v\sqrt{dv}\times:\begin{smallmatrix}+m\\+n\end{smallmatrix}-\frac{mv}{2d}-\frac{mv^2}{8d^2}$ &c.

Then let the coefficients of the correſponding terms be equated, and we ſhall have $m+n=\frac{2}{3}$, and $\frac{m}{2}=\frac{1}{5}$; whence $m=\frac{2}{5}$, and $n=\frac{2}{3}-m=\frac{2}{3}-\frac{2}{5}=\frac{4}{15}$.

Then, by ſubſtituting theſe values of m and n in the aſſumed quantity, we ſhall have $2DP\times\overline{m\times PA+n\times AD}=2DP\times\overline{\frac{2}{5}PA+\frac{4}{15}AD}=\frac{4}{10}DP\times\overline{2PA+\frac{4}{3}AD}$, which is the rule. Or it is $=\frac{4}{15}DP\times\overline{3PA+2AD}$. And this is one of SIR I. NEWTON's rules publiſhed by JONES, WALLIS, and COLSON.

* DEMONSTRATION *of* RULE V.

By rule 2 the ſegment is $=2v\sqrt{dv}\times:\frac{2}{3}-\frac{v}{5d}-\frac{v^2}{28d^2}$, &c.

Suppoſe it equal to $2v\sqrt{mdv-nvv}=2v\sqrt{dv}\times\sqrt{m-\frac{nv}{d}}=2v\sqrt{dv}\times:m^{\frac{1}{2}}-\frac{nv}{2m^{\frac{1}{2}}d}-\frac{n^2v^2}{8m^{\frac{3}{2}}d^2}$ &c. Then, by equating the coefficients, we have $m^{\frac{1}{2}}=\frac{2}{3}$, and $\frac{n}{2m^{\frac{1}{2}}}=\frac{1}{5}$; whence $m=\frac{4}{9}$, and $n=\frac{4}{15}$.

Then, by ſubſtituting theſe values, we get $2v\sqrt{mdv-nvv}=2v\sqrt{\frac{4}{9}dv-\frac{4}{15}vv}=\frac{4}{3}v\sqrt{dv-\frac{3}{5}vv}$, as in the rule.

COROL-

EXAMPLE.

Taking, again, the ſame example;

We have $\frac{4}{3}v\sqrt{\frac{1}{4}CC+\frac{2}{5}vv}=\frac{8}{3}\sqrt{101\frac{1}{3}}=26{\cdot}87915=$ the area nearly.

RULE VI.

If AQ biſect DP, the ſegment $ADBA$ will be very nearly equal to $\overline{4QA+AD}\times\frac{4}{15}DP=2\sqrt{CC+vv}+c\times\frac{4}{15}v=2\sqrt{4dv-3vv}+\sqrt{dv}\times\frac{4}{15}v=2\sqrt{CC+vv}+\sqrt{\frac{1}{4}CC+vv}\times\frac{4}{15}v$; where the letters denote the ſame quantities as in the laſt rule.* [See the laſt figure.]

Ex-

COROLLARY.

But, now, to find what line the quantity $\sqrt{dv-\frac{3}{5}vv}$ repreſents; I conſider that it is greater than AP which is $=\sqrt{dv-vv}$, and therefore I ſuppoſe it to be ſome line AQ meeting DP between D and P. Then $PQ=\sqrt{QA^2-AP^2}=\sqrt{dv-\frac{3}{5}vv-dv+vv}=v\sqrt{\frac{2}{5}}=\frac{v\sqrt{10}}{5}$.

Wherefore, if there be made $DP:PQ::5:\sqrt{10}$, and QA be drawn; the ſegment will be nearly $\frac{4}{3}DP\times AQ$. And this is another of SIR I. NEWTON's Rules.

* DEMONSTRATION *of* RULE VI.

For ſuppoſe the ſegment $2v\sqrt{dv}\times:\frac{2}{3}-\frac{v}{5d}$ &c. to be $=2PD\times\overline{m\times QA+n\times AD}=2v\times\overline{m\sqrt{dv-\frac{3}{4}vv}+n\sqrt{dv}}=2v\sqrt{dv}\times\overline{m\sqrt{1-\frac{3v}{4d}}+n}=2v\sqrt{dv}\times:{+m \atop +n}-\frac{3mv}{8d}$ &c.

Then, equating the coefficients, we get $m+n=\frac{2}{3}$, and $\frac{3m}{8}=\frac{1}{5}$; whence $m=\frac{8}{15}$, and $n=\frac{2}{15}$.

Therefore $2PD\times\overline{m\times QA+n\times AD}=2PD\times\overline{\frac{8}{15}QA+\frac{2}{15}AD}=\frac{4}{15}PD\times\overline{4QA+AD}$, as in the rule. And this likewiſe is a rule given by SIR I. NEWTON.

EXAMPLE.

Taking, ſtill, the ſame example.

We ſhall have $2\sqrt{CC+vv}+\sqrt{\frac{1}{4}CC+vv}\times\frac{4}{15}v=$ 26·8786888 = the area very nearly.

RULE VII.

From the ſquare of the chord of half the arc ſubtract $\frac{5}{7}$ of the ſquare of the verſed ſine of the ſaid half arc, to 7 times the root of the difference add $\frac{4}{3}$ of the ſaid chord, multiply the ſum by the ſaid verſed ſine, and $\frac{4}{75}$ of the product will be the area extremely near.

That is, $\overline{7\sqrt{cc-\frac{5}{7}vv}+\frac{4}{3}c}\times\frac{4}{75}v=\overline{7\sqrt{dv-\frac{5}{7}vv}+\frac{4}{3}\sqrt{dv}}\times\frac{4}{75}v=\overline{7\sqrt{\frac{1}{4}CC+\frac{2}{7}vv}+\frac{4}{3}\sqrt{\frac{1}{4}CC+vv}}\times\frac{4}{75}v=$ the area extremely near: where the letters are the ſame as in the laſt rules.* Ex-

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* DEMONSTRATION.

The ſegment is $=2v\sqrt{dv}\times:\frac{2}{3}-\frac{v}{5d}-\frac{v^2}{28d^2}$ &c.

Suppoſe it $=2v\times\overline{m\sqrt{dv-nvv}+p\sqrt{dv}}=2v\sqrt{dv}\times\overline{p+m\sqrt{1-\frac{nv}{d}}}$ $=2v\sqrt{dv}:\times\begin{matrix}+p\\+m\end{matrix}-\frac{mnv}{2d}-\frac{mn^2v^2}{8d^2}$ &c.

Then, equating the like coefficients, we have $m+p=\frac{2}{3}$, $\frac{mn}{2}=\frac{1}{5}$ and $\frac{mn^2}{8}=\frac{1}{28}$; whence $n=\frac{5}{7}$, $m=\frac{14}{25}$, and $p=\frac{8}{75}$.

Wherefore $2v\times\overline{m\sqrt{dv-nvv}+p\sqrt{dv}}=\frac{4}{75}v\times\overline{7\sqrt{dv-\frac{5}{7}vv}+\frac{4}{3}\sqrt{dv}}$, as in the rule.

COROLLARY.

Suppoſe $\sqrt{dv-\frac{5}{7}vv}=AQ$ [ſee laſt fig.]; then $PQ=\sqrt{QA^2-AP^2}$ $=\sqrt{dv-\frac{5}{7}vv-dv+vv}=v\sqrt{\frac{2}{7}}=\frac{v\sqrt{14}}{7}$.

Wherefore if AQ divide DP in Q ſo that $7:\sqrt{14}::DP:PQ$; then $\overline{7QA+\frac{4}{3}AD}\times\frac{4}{75}DP$ will be nearly equal to the ſegment $ADBA$. And this rule is not only new and very ſimple, but alſo the moſt exact of any approximation that I know of for this purpoſe.

EXAMPLE.

Taking, again, the ſame example as before, where $C=20$, and $v=2$;

We have $\overline{7\sqrt{\tfrac{1}{4}CC+\tfrac{8}{7}vv}+\tfrac{4}{3}\sqrt{\tfrac{1}{4}CC+vv}}\times\tfrac{4}{21}v=$ 26·8787998 = the area very exact.

RULE VIII.

1. Divide the verſed ſine or height by the diameter.

2. Find the quotient in the column of verſed ſines (marked *V. S.*) in the table of circular ſegments at the end of the book.

3. Multiply the number immediately on the right of the verſed ſine, in the table, by the ſquare of the diameter, and the product will be the area.*

Note.

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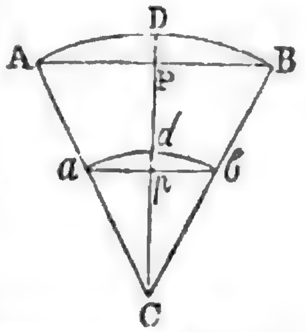
* DEMONSTRATION.



This rule is founded on this property, That ſegments whoſe verſed ſines are as the diameters, will be to each other as the ſquares of the diameters, which is thus proved.

Let $ADBA$, $adba$, be two ſimilar ſegments cut off from the ſimilar ſectors $ADBCA$, $adbCa$, by the chords AB, ab. And draw CD to biſect them.

Then, by ſimilar triangles, $CA : Ca :: CA-DP\ (CP) : Ca-dp\ (Cp) :: DP : dp$. That is, ſimilar ſegments have their verſed ſines as their radiuſes or diameters.

Again, ſince ſimilar ſectors are as the ſquares of the diameters as well as ſimilar triangles as the ſquares of their like ſides, we ſhall have $CA^2 : Ca^2 ::$ ſector $CADB$: ſector $Cadb :: \Delta CAB : \Delta Cab ::$ ſegment $ADBA$ (ſector $CADB - \Delta CAB$) : ſegment $adba$ (ſector $Cadb - \Delta Cab$).

Now, putting $d=$ any diameter, and $v=$ verſed ſine. We ſhall have $d : v :: 1$ (diameter in the table) : $\frac{v}{d}=$ the verſed ſine of a ſimilar ſegment in the table, whoſe area let be called d.

Then $1^2 : dd :: a : add =$ the area of the ſegment whoſe height is v and diameter d, as in the rule.

Note. When the quotient ariſing from the verſed ſine divided by the height hath a remainder or fraction after the fourth place of decimals, having taken the area oppoſite to the ſaid four firſt figures, ſubtract it from the next following area, multiply the remainder by the ſaid fraction, and add the product to the firſt area; and the ſum will be the area for the whole quotient.—This method is uſed when accuracy is required; but for common uſe the fraction may be entirely omitted.

EXAMPLE.

If the diameter be 52 and verſed ſine 2; what is the area?

Firſt $\frac{2}{52} = \frac{1}{26} =$ ·0384$\frac{8}{13}$ the tabular verſed ſine.

Then againſt ·0384 ſtands ·00991672, and the difference between this area and the next ·00995517 is ·00003845, which multiplied by $\frac{8}{13}$ produces ·00002366, which added to ·00991672 gives ·00994038 = the area belonging to the verſed ſine ·0384$\frac{8}{13}$.

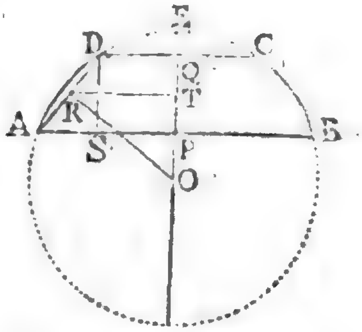
Wherefore ·00994038 × 52 × 52 = 26·87878752 = the area, nearly the ſame as before.

PROBLEM X.

To find the area of the zone or ſpace ABCDA included between two parallel chords AB, CD, and the two arcs AD, CB.

RULE I.

Find the area of each ſegment *ABEA*, *DCED*, and their difference will be that of the zone required.



EXAMPLE I.

Suppoſe the greater chord to be 96, the leſs 60, and the diſtance between them = 26; required the area.

Draw *AD*, and let *DS* be $\perp AP$, $OR \perp AD$, and $RT \perp PQ$; then $QT = TP$, becauſe $AR = RD$; and $TR = \frac{AP+DQ}{2}$. But, in the ſimilar triangles *DAS*, *ROT*, it will be $DS : SA :: RT : TO = \frac{RT \times SA}{DS} = \frac{\overline{PA+DQ} \times \overline{PA-DQ}}{2PQ}$ $= \frac{78 \times 18}{52} = 27$. And hence $OP = OT - TP = 27 - 13 = 14$.

Then $EO = OA = \sqrt{AP^2 + PO^2} = \sqrt{48^2 + 14^2} = 50$ = the radius.

Whence $\left\{ \begin{array}{l} PE = EO - OP = 50 - 14 = 36 \\ QE = EO - OQ = 50 - 40 = 10 \end{array} \right\}$ the two verſed ſines.

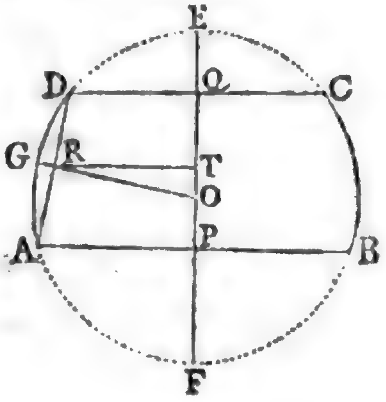
And $\left\{ \begin{array}{l} \frac{36}{100} = \cdot 36 \\ \frac{10}{100} = \cdot 1 \end{array} \right\}$ = the two tab. verſed ſines. Againſt which, in the table, ſtand the areas $\left\{ \begin{array}{l} \cdot 25455055, \\ \cdot 04087528. \end{array} \right.$

$$\text{Then} \left\{ \begin{array}{l} \cdot 25455055 \times 100 \times 100 = 2545 \cdot 5055 \\ \cdot 04087528 \times 100 \times 100 = \underline{408 \cdot 7528} \end{array} \right.$$

their difference $2136 \cdot 7527$ = the area of the zone required.

EXAMPLE II.

Suppoſe the greater chord *AB* to be 20, the leſs *DC* 15, and their diſtance *PQ* $17\frac{1}{2}$. Required the area of the zone *ABCD*.



Then, as in the laſt example, $TO = \frac{\overline{PA+DQ} \times \overline{PA-DQ}}{2PQ} = \frac{35 \times 5}{2 \times 2 \times 35} = \frac{5}{4}$. Which being leſs than TP $(8\frac{3}{4})$ half the diſtance of the chords, ſhews that the center falls within the zone; and then it is evident that the whole circle diminiſhed by the ſum of the two ſegments AFB, DEC, will give the zone.

Now, $OP = OT - TO = 8\frac{3}{4} - \frac{5}{4} = 7\frac{1}{2}$. Hence $AO = \sqrt{\overline{OP^2 + PA^2}} = \sqrt{\left.\frac{15}{2}\right|^2 + 10^2} = \frac{25}{2} =$ the radius; therefore the diameter is 25. And $OQ = QT + TO = 8\frac{3}{4} + 1\frac{1}{4} = 10$.

$$\text{Whence } \begin{cases} QE = EO - OQ = 12\frac{1}{2} - 10 = 2\frac{1}{2}, \\ PF = FO - OP = 12\frac{1}{2} - 7\frac{1}{2} = 5. \end{cases}$$

$$\text{Then } \left\{ \begin{array}{l} \frac{2\frac{1}{2}}{25} = \frac{5}{50} = \frac{1}{10} = {\cdot}1 \\ \frac{5}{25} = \frac{1}{5} = {\cdot}2 \end{array} \right\} \text{the tab. verſ. ſines.}$$

$$\text{Againſt which ſtand the areas } \begin{cases} {\cdot}04087528, \\ {\cdot}11182380. \end{cases}$$

$$\text{Then } \begin{cases} {\cdot}04087528 \times 25 \times 25 = \text{the ſegment } DEC, \\ {\cdot}11182380 \times 25 \times 25 = \text{the ſegment } AFB, \end{cases}$$

whoſe ſum is $\cdot 15269908 \times 25^2$.

But $\cdot 78539816 \times 25 \times 25 =$ the whole circle.

Therefore $25^2 \times \overline{\cdot 78539816 - \cdot 15269908} = 25^2 \times \cdot 63269908 = 395\cdot 436925 =$ the zone $ABCD$ required.

RULE II.

To the area of the trapezoid $APQD$ add that of the ſmall ſegment $AGDRA$, and double the ſum will be the area of the zone $ABCD$ required.—And this rule is eaſier than the other as only one ſegment is to be found.

EXAMPLE.

Suppoſe the ſame things as in the laſt example.

Then, drawing the radius ORG, we ſhall have $OR = \sqrt{\overline{RT}^2 + \overline{TO}^2} = \sqrt{\left|\frac{DQ+PA}{2}\right|^2 + \overline{TO}^2} = \sqrt{\left|\frac{35}{4}\right|^2 + \left|\frac{5}{4}\right|^2} = \frac{5}{4}\sqrt{7^2 + 1^2} = \frac{5}{4}\sqrt{50} = \frac{25}{4}\sqrt{2} = 8·838834765$. Hence $RG = GO - OR = 12·5 - 8·838834765 = 3·661165$ the verſed ſine.

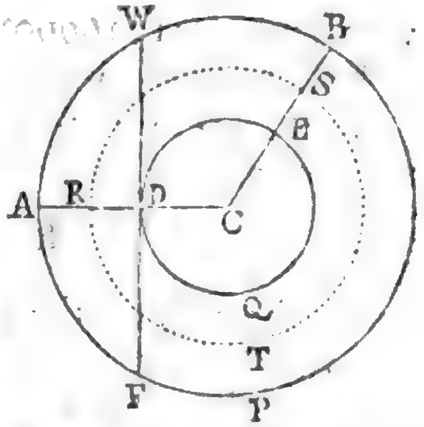
Therefore $\frac{3·661165}{25} = ·1464466$ = the tabular verſed ſine; the tabular area for which is ·07134954.

Then $·07134954 \times 25 \times 25 = 44·5934625$ = the ſegment $AGDRA$.

But $\frac{DQ+PA}{2} \times PQ = \frac{35}{4} \times \frac{35}{2} = \frac{1225}{8} = 153·125$ = the trapezoid $APQD$.

Therefore $153·125 + 44·5934625 = 197·7184625$ = the ſemi-ſegment $APQDGA$. The double of which is $395·436925$ = the zone $ABCD$, the ſame as before.

PROBLEM XI.



To find the area of the ring included between the circumferences ABPA, DEQD of two concentric circles.

RULE I.

Multiply the ſum of the diameters by the difference of the diameters, and the product drawn into ·7854 will be the area of the ring required.*

Ex-

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* DEMONSTRATION.

* For, ſince the ring is evidently equal to the difference of the two circles, if the diameters be called D, d, and $·785398$ &c. $= a$; he ring will be $= aD^2 - ad^2 = a \times \overline{D+d} \times \overline{D-d}$. $Q.E.D.$

Co-

EXAMPLE.

If the diameters of two concentric circles be 10 and 6, what will be the area of the ring included between their circumferences?

$\cdot 7854 \times \overline{10+6} \times \overline{10-6} = \cdot 7854 \times 16 \times 4 = 50{\cdot}2656 =$ the area required.

RULE II.

Multiply half the ſum of the circumferences by half the difference of the diameters, and the product will be the area.*

EXAMPLE.

Taking the ſame example as before, we ſhall have

$$\text{Firſt } 3{\cdot}1416 \times \left\{ \begin{matrix} 10 = 31{\cdot}416 \\ 6 = 18{\cdot}8496 \end{matrix} \right\} = \text{the circumferences.}$$

$$\text{Then } \frac{31{\cdot}416 + 18{\cdot}8496}{2} \times \frac{10-6}{2} = \frac{50{\cdot}2656}{2} \times \frac{4}{2} = 50{\cdot}2656 =$$

the area the ſame as before.

COROLLARY I.

Whence, if there be made DW perpendicular to the radius CDA, the ring will be equal to the circle whoſe radius is DW; or equal to the circle whoſe diameter is FW a tangent to the leſs circle, and terminated by the greater.

COROLLARY II.

Or the ring is equal to an ellipſe whoſe axes are $D+d$ and $D-d$. As will appear by comparing the above rule with the rule for an ellipſe to be hereafter given.

* DEMONSTRATION *of* RULE II.

For the circumferences C, c, are equal to $4aD$, $4ad$; therefore $a \times \overline{D+d} = \frac{C+c}{4}$; which ſubſtituted in the laſt rule, gives $a \times \overline{D+d} \times \overline{D-d} = \frac{C+c}{4} \times \overline{D-d} = \frac{C+c}{2} \times \frac{D-d}{2}$, as in the rule. Q.E.D.

And it is evident that the ſame rule will ſerve for a part of the ring cut off by two radiuſes, uſing the lengths of the included curves for C and c.

Note. This rule will ſerve, alſo, for any part $ABEDA$ of the ring included between the parts AD, BE, of two radiuſes, by uſing half the ſum of their intercepted arcs for half the ſum of the circumferences.

Thus, if the length of the arc AB be 15; then, by reaſon of the ſimilarity of the arcs AB, DE, it will be $10:6::15:\frac{15\times6}{10}=3\times3=9$ = the arc DE.

Whence $\frac{15+9}{2}\times\frac{10-6}{2}=12\times2=24=$ the area of the part $ABEDA$.

RULE III.

Multiply the perpendicular breadth of the ring, that is, the difference of the radiuſes, by the circumference RST (or part RS for the part $ABEDA$) having the ſame center with, and equally diſtant from the bounding arcs.*

EXAMPLE.

Taking, ſtill, the ſame example; it will be

Firſt $DA=AC-CD=5-3=2=$ the diſtance of the circumferences.

And $RC=CD+DR=CD+\frac{1}{2}DA=3+1=4=$ the radius of the middle arc.

Hence $3\cdot1416\times8=25\cdot1328=$ the middle circumference.

Wherefore $25\cdot1328\times2=50\cdot2656=$ the area the ſame as before.

And for the part $ABEDA$, we have $AC:CR::AB:SR=\frac{AB\times RC}{CA}=\frac{15\times4}{5}=3\times4=12$.

Then $AD\times RS=2\times12=24=$ the area.

PRO-

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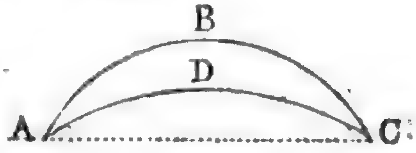
* DEMONSTRATION.

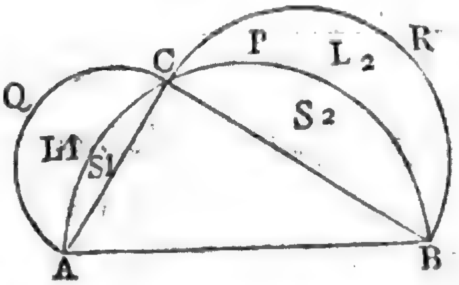
For this circumference, being equally diſtant from the other two, will be equal to half their ſum. Wherefore &c.

PROBLEM XII.

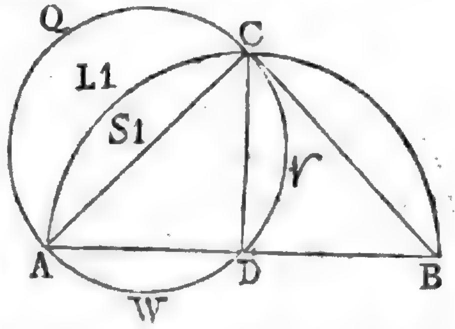
To find the areas of lunes.



Lunes, *lunulæ*, or little-moons, are ſpaces included between the interſecting arcs of two excentric circles, as the lune $ABCDA$; which is, evidently, equal to the difference of the ſegments $ABCA$, $ADCA$.



If ABC be a triangle right-angled at C, and if ſemicircles be deſcribed on the three ſides as diameters; Then the triangle T (ABC) will be equal to the ſum of the two lunes $L1$, $L2$.—For the greateſt ſemicircle is equal to the ſum of the other two; from the greateſt ſemicircle take the ſegments $S1$, $S2$, and there will remain the $\triangle T$; from the two leſs ſemicircles take, alſo, the two ſegments $S1$, $S2$, and there will remain the two lunes $L1$, $L2$; wherefore $T = L1 + L2$.



Whence, if the two ſides AC, CB, of the triangle be equal to each other; the two lunes will, alſo, be equal, and each lune $L1$ equal to the $\triangle ACD$; and therefore the ſegment $S1$ = ſemicircle $AQCA - \triangle ACD = 2$ times the ſeg. $AWDA = 2$ times the ſeg. $DVCD$.

PRO-

PROBLEM XIII.

To find the areas of irregular polygonal figures, ſuch as moſt pieces of land generally are.

RULE.

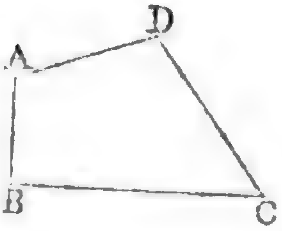
Divide them into triangles and trapeziums, and add their areas together. But for the beſt methods of doing this, I muſt refer the reader to the latter part of this work, in which ſurveying is particularly treated of.

*SECT. II.

A promiſcuous Collection of Queſtions concerning Areas.

QUESTION I.

IN the trapezium *ABCD* are given $AB = 6\frac{1}{2}$, $BC = 15\frac{3}{5}$, $CD = 12$, and $DA = 9$, alſo *B* a right angle; to find the area of the trapezium.



Firſt $\frac{6\frac{1}{2} \times 15\frac{3}{5}}{2} = \frac{13}{4} \times \frac{78}{5} = \frac{1014}{20} = 50{\cdot}7$ = the area of the triangle *ABC*.

But $\sqrt{AB^2 + BC^2} = \sqrt{\overline{\frac{13}{2}}\Big|^2 + \overline{\frac{78}{5}}\Big|^2} = 16{\cdot}9 = AC$; and $\sqrt{\frac{37{\cdot}9}{2} \times \frac{4{\cdot}1}{2} \times \frac{13{\cdot}9}{2} \times \frac{19{\cdot}9}{2}} = \frac{1}{4}\sqrt{37{\cdot}9 \times 4{\cdot}1 \times 13{\cdot}9 \times 19{\cdot}9} = \frac{1}{4}\sqrt{42982{\cdot}4279} = 51{\cdot}8305098$ = the area of the triangle *ADC*.

Whence $50{\cdot}7 + 51{\cdot}8305098 = 102{\cdot}5305098$ = the area of the trapezium required.

QUES-

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[*] Some of the queſtions in this ſection are taken from other books; but the methods of ſolution are, generally, different from thoſe uſed in the books from which they were taken, they being there moſtly ſolved by an analytical proceſs. And I have conſtructed thoſe of which the conſtructions do not appear to be ſelf-evident.

QUESTION II.

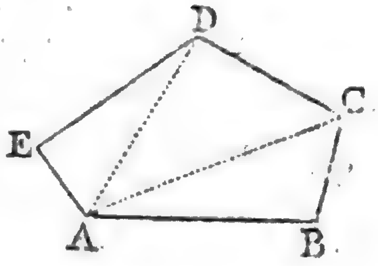
One morning in may, I went to ſurvey
As ſoon as bright ſol I eſpi'd;
I meaſur'd around a four-corner'd ground,
I'th' margin's the length of each ſide:
The angle at *B* together with *D*,
An hundred and 80 degrees,
The meadow's content is all that I want;
Aſſiſt me, kind ſirs, if you pleaſe.

Chains
$AB = 15\cdot6$
$BC = 13\cdot2$
$CD = 10\cdot0$
$DA = 26\cdot0$

Since a trapezium can be inſcribed in a circle when the ſum of two of its oppoſite angles is 180°; therefore, by rule 5 of prob. 3 of ſect. 1. ($32\cdot4$ being half the perimeter) $\sqrt{\overline{32\cdot4-15\cdot6}\times\overline{32\cdot4-13\cdot2}\times\overline{32\cdot4-10}\times\overline{32\cdot4-26}} = \sqrt{16\cdot8\times19\cdot2\times22\cdot4\times6\cdot4} = 215\cdot04$ ſquare chains $= 21\cdot504$ acres $= 21$ a. 2 r. $0\cdot64$ perches $=$ the area required.

QUESTION III.

In the pentangular field *ABCDE* are given $AB = 14$, $BC = 7$, $CD = 10$, $DE = 12$, $EA = 5$, and the diagonal $AC = 17$ chains, alſo *E* a right angle; required the area.



Firſt $AD = \sqrt{DE^2 + EA^2} = \sqrt{12^2+5} = 13$.

Then $AE \times \frac{1}{2} ED = 5 \times 6 = 30 =$ the area of the triangle *ADE*.

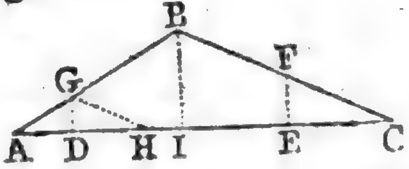
And $\sqrt{20\times3\times7\times10} = 10\sqrt{42} = 64\cdot80740698 = \triangle ADC$.

Alſo $\sqrt{19\times2\times5\times12} = 2\sqrt{570} = 47\cdot74934554 = \triangle ABC$.

Wherefore the ſum of the three is $142\cdot55675252$ ſquare chains $= 14$ a. $1\cdot0227$ r. $=$ the area required.

QUESTION IV.

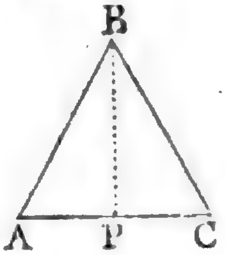
Given the baſe $AC = 32$, $AD = 5$, $EC = 9$, the perpendicular $EF = 4$, and the perpendicular $DG = 3$; required the area of the triangle ABC.



Draw GH parallel to BC; then, by the ſimilar triangles GDH, FEC, $FE = 4 : EC = 9 :: GD = 3 : DH = \frac{27}{4}$, and hence $AH = \frac{47}{4}$; again, from the ſimilar triangles AGH, ABC, $AH = \frac{47}{4} : AC = 32 :: DG = 3$: the perpendicular $IB = \frac{384}{47}$. Hence $\frac{1}{2} AC \times IB = 16 \times \frac{384}{47} = 130\frac{34}{47} = 130{\cdot}7234\frac{2}{47}$ = the area of the triangle required.

QUESTION V.

What is the ſide of that equilateral triangle, whoſe area coſt as much paving at 8d. a foot, as the palliſading the three ſides did at a Guinea a yard?

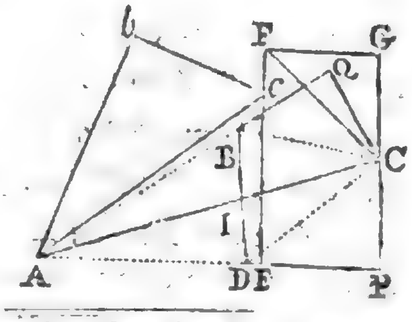


The ſides are 7s. a foot, and the area $\frac{2}{3}$s. a foot. And that the produces may be equal, the quantities muſt be inverſly as the prices; but, by rule 2 page 80, $\frac{1}{4} BC^2 \sqrt{3}$ = the area; therefore $\frac{2}{3} : 7 :: 3BC : \frac{1}{4} BC^2 \sqrt{3} :: \frac{3 \times 4}{\sqrt{3}} : BC = \frac{3 \times 4 \times 7 \times 3}{2\sqrt{3}} = 42\sqrt{3} = 72{\cdot}7461339$ = the ſide required.

QUESTION VI.

Surveying a field, I found the four ſides thereof to be 10, 9, 7, and 6 chains, in a ſucceſſive order: I likewiſe, at the two extremes of the longeſt ſide, took the bearings of the oppoſite angles, which were *N. E.*

by *E.* and *N. W.* Hence the content of the field is required.



From the given bearings of the angles *C* and *D* from *A* and *B*, it appears that the diagonals *AC*, *BD*, make the angle *AID* at their interſection of 7 points or $78\frac{3}{4}$ deg. wherefore, by rule 3 of problem 3 of ſection 1, we have $10^2+7^2-9^2+6^2\times\frac{1}{4}$ tang. $78\frac{3}{4}° = 32\times\frac{1}{4}\times5{\cdot}0273395 = 8\times5{\cdot}0273395 = 40{\cdot}218716$ ſquare chains $= 4$a. $3{\cdot}499456$ perches = the content required.*

QUESTION VII.

Given two ſides of an obtuſe-angled triangle, which are 20 and 40 poles; required the third ſide, that the triangle may contain juſt an acre of land.

Now

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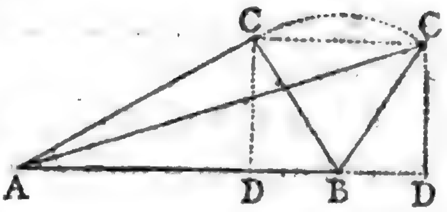
* CONSTRUCTION.

With two of the given lines Ab (10) and bc (6) make a right angle b; and with the other two complete the trapezium $AbcD$: In the perpendicular Ec, produced, take $EF = 8{\cdot}937492 =$ the double of the area ($40{\cdot}218716$) divided by AD (9); with the center F and radius equal to ($6\frac{2}{3}$) a fourth proportional to DA, Ab, bc, deſcribe an arc meeting, in C, another arc deſcribed with the center D and radius Dc; then join D, C, and with the other two given lines ($= cb$ and bA) complete the trapezium $ABCD$, and it is done.

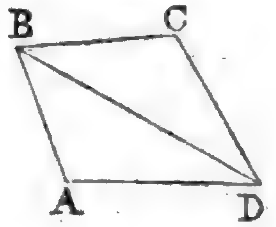
For, having drawn cA, AC, and CF, and let fall the perpendiculars CP upon AD, CQ upon AB, and FG upon PCG; ſince $AD^2 + DC^2 + 2AD\times DP = AC^2 =) CB^2 + BA^2 + 2AB\times BQ$, and $cD^2 + DA^2 + 2AD\times DE = (Ac^2 =) cb^2 + bA^2$, by taking theſe latter quantities from the former, it follows that $2AD\times EP = (2AD\times DP - 2AD\times DE =) 2AB\times BQ$, and conſequently $BQ : FG\ (EP) :: DA : AB :: BC : CF$ (by the conſtruction); whence the triangles BQC, CGF, are ſimilar, and therefore $QC : CG :: BC : CF ::$ (by the conſtruction) $DA : AB$; and hence $QC\times AB = AD\times CG$; and, by adding $CP\times DA$ to each, we have $CP\times DA + CQ\times AB$ ($=$ twice the area $ABCD$) $= CP\times DA + AD\times CG = EF\times DA =$ (by the conſtruction) twice the given area.



Now the area $160 \div 20$ $(\frac{1}{2}AB) = 8 =$ the perpendicuar CD; but $\sqrt{BC - CD^2} = \sqrt{20^2 - 8^2} = DB$; hence $AD = AB \pm BD = 40 \pm \sqrt{20^2 - 8^2}$, and $AC = \sqrt{CD^2 + DA^2} = \sqrt{8^2 + 40^2 + 20^2 - 8^2 \pm 80\sqrt{20^2 - 8^2}} = 4\sqrt{125 \pm 20\sqrt{21}} = 4 \times \sqrt{105} \pm 2\sqrt{5} = 58\cdot87634686$ and $23\cdot09925922$, either of which may be the ſide required.

QUESTION VIII.

Given the four ſides $AB = 9$, $BC = 10$, $CD = 11$, and $DA = 12$, and the angle $BDC = 30°$, formed by the diagonal BD and the ſide DC; required the area of the trapezium.



Firſt, by trigonometry, $BC : CD :: \frac{1}{2}(\text{s.}\angle BDC) : \frac{DC}{2CB} = \frac{11}{20} = \cdot55 =$ the ſine of the angle $DBC = 33° 22'$; whence $180° - 30° + 33° 22' = 180° - 63° 22' = 116° 38' = \angle C$; and $\text{s.}\angle BDC : \text{s.}\angle C :: CB : BD = 10 \times \cdot8938936 \times 2 = 17\cdot877872$.

Then $BC \times CD \times \frac{1}{2}\,\text{s.}\angle C = 10 \times 11 \times \cdot4469468 = 49\cdot164148 =$ the area of the triangle BCD.

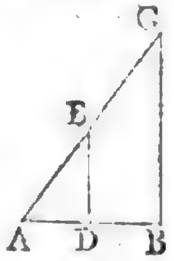
And $\sqrt{19\cdot438936 \times 10\cdot438936 \times 7\cdot438936 \times 1\cdot561064} = 48\cdot54337 =$ the area of the triangle ABD.

Wherefore $49\cdot164148 + 48\cdot54337 = 97\cdot707518 =$ the area of the whole trapezium.

QUESTION IX.

If from the right-angled triangle ABC, whoſe baſe is 12, and perpendicular 16 feet, be cut off, by a line

DE parallel to the perpendicular, a triangle whoſe area is 24 ſquare feet; what are the ſides of this triangle?



The triangles *ABC*, *ADE*, are ſimilar, and the areas of ſimilar triangles are as the ſquares of their like ſides; but 12×8 = 96 is the area of the triangle *ABC*,

$$\therefore \sqrt{96}:\sqrt{24}::\left\{\begin{array}{l}16:16\sqrt{\tfrac{24}{96}}=16\sqrt{\tfrac{1}{4}}=16\times\tfrac{1}{2}=8=ED\\ 12:12\sqrt{\tfrac{24}{96}}=12\sqrt{\tfrac{1}{4}}=12\times\tfrac{1}{2}=6=DA\end{array}\right\},$$

and $AE=\sqrt{AD^2+DE^2}=\sqrt{6^2+8^2}=\sqrt{100}=10$.

QUESTION X.

If from a triangle whoſe ſides are 13, 14, 15, be cut off by a line drawn parallel to the longeſt ſide, an area of 24; what are the lengths of the ſides including that area?

Firſt, $\frac{13+14+15}{2}=\frac{42}{2}=21$ = half the ſum of the ſides, from which each ſide being taken, leaves 6, 7, 8; therefore $\sqrt{21\times6\times7\times8}=\sqrt{3^2\times7^2\times4^2}=3\times4\times7=84$ = the area of the given triangle. Then, as in the laſt,

$$\sqrt{84}:\sqrt{24}::\left\{\begin{array}{l}13:13\sqrt{\tfrac{2}{7}}=\tfrac{13}{7}\sqrt{14}\\ 14:14\sqrt{\tfrac{2}{7}}=2\sqrt{14}\\ 15:15\sqrt{\tfrac{2}{7}}=\tfrac{15}{7}\sqrt{14}\end{array}\right\}=\text{the three ſides}$$

incloſing the area cut off.

QUESTION XI.

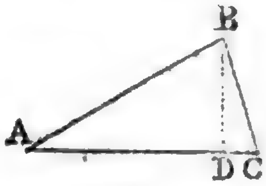
If two ſides of a triangle, whoſe area is $\sqrt{10800}=60\sqrt{3}$, be 20 and 12, what is the third ſide?

From the end of one of the given ſides let fall the perpendicular *CD* upon the other ſide produced.

Then (fig. to queſt. 7) $CD = \frac{60\sqrt{3}}{\frac{1}{3}\times 12} = 10\sqrt{3}$, and $DB = \sqrt{BC^2 - CD^2} = \sqrt{20^2 - 3\times 10^2} = 10$; hence $DA = AB \pm BD = 22$ or 2; and $CA = \sqrt{AD^2 + DC^2} = \sqrt{22^2 \text{ or } 2^2 + 3\times 10^2} = \sqrt{784}$ or $\sqrt{304} = 28$ or $4\sqrt{19}$.

QUESTION XII.

Given the area $= 144$, the baſe $AC = 24$, and one angle $BAC = 30°$, at the baſe; to find the ſides AB, BC.



Firſt, $144 \div 12$ $(\frac{1}{2}AC) = 12 =$ the perpendicular BD.

Then, as $\frac{1}{2} = \text{s.}\angle A\,30° : 1 = \text{s.}\angle D :: 12 = DB : 24 = BA$.

But $DA = \sqrt{AB^2 - BD^2} = \sqrt{24^2 - 12^2}$, and $DC = CA - AD = 24 - \sqrt{24^2 - 12^2}$.

Wherefore $BC = \sqrt{CD^2 + DB^2} = 24\sqrt{2 - \sqrt{3}} = 12 \times \sqrt{6} - \sqrt{2} = 12{\cdot}423314184$.

Or, ſince AB is $= AC$, the $\angle C$ will be $=$ the $\angle B = \frac{180-30}{2} = \frac{150}{2} = 75°$; hence $\angle DBC = 15°$; and therefore as radius $= 1 :$ ſec. $15° = 1{\cdot}0352762 :: BD = 12 : 12 \times 1{\cdot}0352762 = 12{\cdot}4233144 = BC$.

Note. It is pretty evident, that this triangle will be conſtructed by drawing *AB* to make with the given baſe *AC* an angle of 30°, and meeting with a line drawn parallel to and at the diſtance of the perpendicular *BD* from *AC*, in *B* the vertex of the triangle.

QUESTION XIII.

Given the area 1012, and ratio of the ſides of a triangle, viz. *AC* to *CB*, as 4 to 3, and *CB* to *BA* as 2 to 3; to find the ſides.

It is evident that the ſides *AC*, *CB*, *BA*, are in the ratio of the three numbers 8, 6, 9, reſpectively, and therefore a triangle whoſe ſides are 8, 6, 9, will be ſimilar to the triangle propoſed; but ſimilar triangles are as the ſquares of their like ſides, and $\sqrt{\frac{23}{2} \times \frac{5}{2} \times \frac{7}{2} \times \frac{11}{2}} = \frac{1}{4}\sqrt{8855}$ = the area of the triangle whoſe ſides are 8, 6, 9; therefore $\frac{1}{2}\sqrt[4]{8855} : \sqrt{1012} ::$

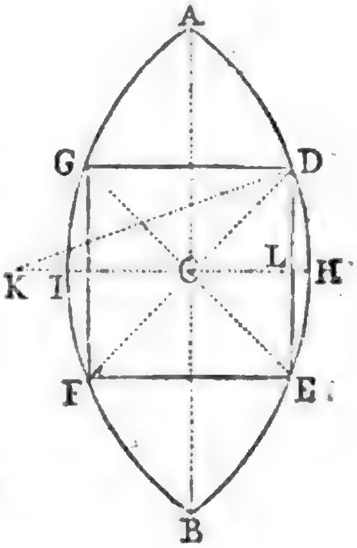
$$\begin{cases} 8 : 32\sqrt[4]{\frac{23\times11}{35}} = \frac{32}{35}\sqrt[4]{10847375} = 52{\cdot}47024 = AC. \\ 6 : 24\sqrt[4]{\frac{23\times11}{35}} = \frac{24}{35}\sqrt[4]{10847375} = 39{\cdot}35268 = CB. \\ 9 : 36\sqrt[4]{\frac{23\times11}{35}} = \frac{36}{35}\sqrt[4]{10847375} = 59{\cdot}02902 = BA. \end{cases}$$

QUESTION XIV.

The diſtance of the centers of two circles, whoſe diameters are each 50, being given equal to 30; it is required to find the area of the ſpace incloſed by their circumferences, and that of a ſquare inſcribed in the ſaid ſpace?

Conſtruction.

Having deſcribed the circumferences including the common ſpace *AHBI*, *IH* being the common part of both the diameters, biſect *IH* in *C*, and through *C* draw *FCD*, *GCE*, making each half a right angle with *IH*, and meeting the arcs in *D*, *E*, *F*, *G*, the four angular points of the ſquare.



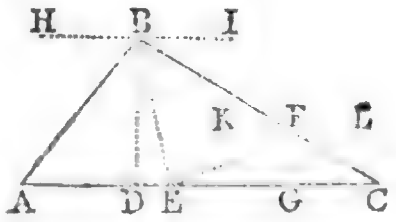
Calculation.

If K be the center of one of the circles, then $HK-KC = 25-15=10=CH=CI=$ the height of each ſegment; and $\frac{CH}{2HK}=\frac{10}{50}=\cdot2=$ the tab. verſed ſine; againſt which, in the table of circular ſegments, ſtands the area $\cdot11182380$; therefore $\cdot1118238\times50\times50\times2=559\cdot119$ = the two ſegments or common ſpace $AHBIA$.

Again, drawing KD, we ſhall have as $DK=25$: $KC=15$ :: s. $\angle C=135^\circ$ or 45° : s. $\angle CDK=25^\circ\ 6\frac{1}{4}'$; hence $180^\circ-135^\circ-25^\circ\ 6\frac{1}{4}'=180^\circ-160^\circ\ 6\frac{1}{4}'=19^\circ\ 53\frac{3}{4}'=\angle K$; then as radius = s. $\angle L$: s. $\angle K=19^\circ\ 53\frac{3}{4}'$ :: $KD=25$: $DL=8\cdot553925$, the double of which is $DE=17\cdot10785=$ the ſide of the ſquare. And therefore $17\cdot10785\times17\cdot10785=292\cdot67853=$ the area of the ſquare.

QUESTION XV.

Given the baſe $AC=15$, the area 45, and the ratio of AB to BC as 2 to 3; to find the ſides AB, BC, of the triangle?



Firſt, the area $45\div7\frac{1}{2}$ $(\frac{1}{2}AC)=6=$ the perpendicular BD. Then take AE : EC :: AB : BC, that is $5=2+3:15=AC::\left\{\begin{matrix}2:6=AE\\3:9=EC\end{matrix}\right\}$; and ſuppoſe $FB=BA$, as alſo the lines BE, EF, and the perpendicular FG to be drawn; and we ſhall have the two triangles ABE, EBF, equal to each other in every reſpect; but $\frac{1}{2}BD\times\left\{\begin{matrix}AE\\EC\end{matrix}\right\}=3\times\left\{\begin{matrix}6=18=\triangle ABC\\9=27=\triangle EBC\end{matrix}\right\}$; and the $\triangle EBC-\triangle BEF\ (\triangle ABE)$

=

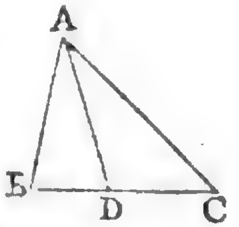
$=27-18=9=\triangle EFC$; hence $9\ (\triangle EFC)\div 4\frac{1}{2}\ (\frac{1}{2}EC)$ $=2=$ the perpendicular FG; but $GE=\sqrt{EF^2-FG^2}$ $=\sqrt{6^2-2^2}=\sqrt{32}$, and $GC=CE-EG=9-\sqrt{32}$; whence, by ſimilar triangles, $FG:GC::BD:DC=27-3\sqrt{32}$, and, conſequently, $DA=AC-CD=3\sqrt{32}-12$; wherefore, laſtly, $AB=\sqrt{BD^2+DA^2}=\sqrt{6^2+\overline{3\sqrt{32}-12}|^2}=$ $3\sqrt{2^2+\overline{\sqrt{32}-4}|^2}=6\sqrt{13-8\sqrt{2}}=7\text{·}79146$, and, as $2:3::AB:BC=9\sqrt{13-8\sqrt{2}}=11\text{·}68719$.

From the foregoing ſolution the following conſtruction is evident.

Having drawn HI parallel to AC, and at ſuch diſtance from it as is expreſſed by the perpendicular BD, found from the given area and baſe, and taken AE to EC in the given proportion of AB to BC; with the center E and radius EA deſcribe an arc meeting in F, with a line KL drawn parallel to AC and at ſuch diſtance from it as is denoted by a third proportional to CE, $CE-EA$, and BD; then through F draw CB meeting with HI in B, and join B, A.

QUESTION XVI.

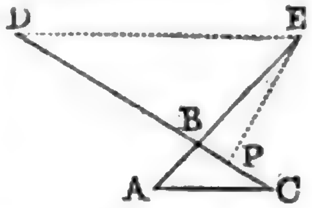
Given the ſegments $BD=10$, and $DC=14$ of the baſe BC made by a line AD biſecting the vertical angle A, and the ſum of the ſides $BA+AC=48$; to find the area.



If a line biſect any angle of a triangle and cut the oppoſite ſide, it is known that the ſegments of that ſide are to each other as the other ſides adjacent to them. Wherefore $24=BC:48=BA+AC::\left\{\begin{matrix}10=BD:20=BA,\\14=DC:28=CA;\end{matrix}\right\}$ hence the area is $\sqrt{36\times8\times12\times16}=6\times2\times2\times4\sqrt{2\times3}=96\sqrt{6}=$ $235{\cdot}15100154$.

QUESTION XVII.

Having given the continuations of the two ſides of the vertical angle of a given triangle, to find the area of the triangle formed by the ſaid continuations and a line connecting their extremities.



That is, Given $AB=2$, $BC=3$, $CA=4$; and $AE=7$, $CD=10$, or $BE=5$, $BD=7$; required DE and the area of the triangle DBE.

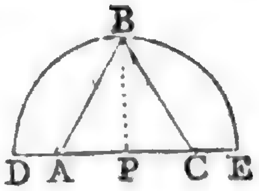
Firſt, $\sqrt{\frac{9}{2}\times\frac{1}{2}\times\frac{3}{2}\times\frac{5}{2}}=\frac{3}{4}\sqrt{15}=$ the area of the triangle ABC. But ſince the vertical angles at B are equal, and triangles having one angle in each equal, are as the rectangles of the ſides about the equal angles; we ſhall have $6:35::\frac{3}{4}\sqrt{15}:\frac{35}{8}\sqrt{15}=16{\cdot}944305=$ the area of the triangle BDE.

Then, having drawn EP perpendicular to BC, $\frac{35}{8}\sqrt{15}\div\frac{7}{2}\,(\frac{1}{2}BD)=\frac{5}{4}\sqrt{15}=EP$; but $\sqrt{BE^2-EP^2}=$ $\sqrt{25-\frac{15}{16}\times25}=\sqrt{\frac{25}{16}}=\frac{5}{4}=PB$, and $PB+BD=7+\frac{5}{4}=\frac{33}{4}=$ DP; hence $\sqrt{DP^2+PE^2}=\sqrt{\overline{\tfrac{33}{4}}^2+15\times\overline{\tfrac{5}{4}}^2}=$ $\frac{1}{4}\sqrt{33^2+15\times5^2}=\frac{1}{4}\sqrt{1464}=\frac{1}{2}\sqrt{366}=9{\cdot}565563235=$ DE.

QUESTION XVIII.

Given the area (100) of the equilateral triangle ABC, whoſe baſe falls on the diameter and vertex in the middle of the arc of a ſemicircle; required the diameter DE of the ſemicircle?

Since

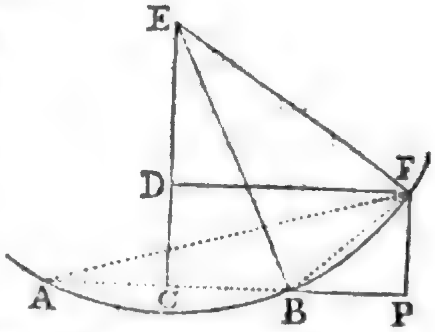


Since the area of an equilateral triangle is equal to the ſquare of the ſide drawn into $\frac{1}{4}\sqrt{3}$, we ſhall have the ſide $AB=\sqrt{\frac{4\times100}{\sqrt{3}}}$, and the half of it $AP=\sqrt{\frac{100}{\sqrt{3}}}=\frac{10}{\sqrt[4]{3}}$; whence the radius of the circle or perpendicular of the triangle BP is $=$ area $\div AP=100\div\frac{10}{\sqrt[4]{3}}=10\sqrt[4]{3}$, the double of which is $20\sqrt[4]{3}=26{\cdot}32148026=$ the diameter required.

QUESTION XIX.

The area of a triangular meadow is 100 perches or ſquare poles, and the length of one of its ſides is 40 poles, which, alſo, is a mean proportional between the other two ſides; from which it is required to find the unknown fences?

Conſtruction.



Make $AB=40=$ the baſe or given ſide, on the middle C of which raiſe CD perpendicular and equal to $(\frac{200}{40}=)$ 5, the perpendicular height of the triangle. Then ſince the rectangle of two ſides of a triangle drawn into the ſine of their included angle is equal to double the area, we ſhall have $\left(\frac{200}{40\times40}=\right)\frac{1}{8}$ for the ſine of the vertical angle; therefore draw BE making the angle CBE equal to the complement of that angle, meeting CD, produced, in E; ſo will E be the center of the circle circumſcribing the triangle; therefore, with the center *E* and radius *EB* or *EA*, deſcribe an arc *ABF* meeting *DF*, drawn parallel to *AB*, in *F*; then draw *AF*, *FB*, and it is evident that *ABF* will be the triangle required.

Calculation.

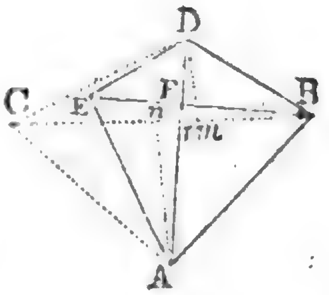
Draw *FP* perpendicular to and meeting *AB*, produced, in *P*. Then, ſince $\frac{1}{8}$ is the s. of $\angle E$, it will be as $\frac{1}{8} : 1 = \text{s.}\,\angle C :: 20 = CB : 160 = BE$; or thus, by geometry, $\frac{AF \times FB}{2FP} = \frac{40 \times 40}{10} = 160 = BE$. Again, $CE = \sqrt{EB^2 - BC^2} = \sqrt{160^2 - 20^2} = 20\sqrt{63} = 60\sqrt{7}$; hence $DE = EC - CD = 60\sqrt{7} - 5$, and $DF = \sqrt{FE^2 - ED^2} = 5\sqrt{15 + 24\sqrt{7}}$; but $BP = DF - CB = 5\sqrt{15 + 24\sqrt{7}} - 20$, and therefore $BF = \sqrt{FP^2 + PB^2} = 10\sqrt{8 + 6\sqrt{7} - 2\sqrt{15 + 24\sqrt{7}}} = 24{\cdot}80833$, and conſequently $FA = \frac{AB^2}{BF} = \frac{40 \times 40}{24{\cdot}80833} = 64{\cdot}49447$.

QUESTION XX.

The four ſides of a field, whoſe diagonals are equal to each other, are known to be 25, 35, 31, and 19 poles, in a ſucceſſive order; from whence the content of the field is required?

Conſtruction.

Make *AB* and *AC* perpendicular to one another, and each equal to one of the given ſides, as 35; with the centers *B*, *C*, and radiuſes equal to the two remaining oppoſite ſides 25 and 31, deſcribe two arcs, interſect-



ing in D, draw AD, and BE perpendicular and equal to it; draw AE and DE, and $ABDE$ will be the trapezium.*

Calculation.

Draw BC meeting AD in r, upon which let fall the perpendiculars An, Dm. Then $BC = \sqrt{BA^2 + AC^2} = 35\sqrt{2}$, and $An = nB = \frac{1}{2} BC = \frac{35\sqrt{2}}{2} = \frac{35}{\sqrt{2}}$; again, $mB = \frac{DB^2 + BC^2 - CD^2}{2BC} = \frac{25^2 + 2 \times 35^2 - 31^2}{70\sqrt{2}} = \frac{2114}{70\sqrt{2}} = \frac{151\sqrt{2}}{10}$, and $mn = nB - Bm = \overline{\frac{35}{2} - \frac{151}{10}} \times \sqrt{2} = \frac{12}{5}\sqrt{2}$; but $mD = \sqrt{DB^2 - Bm^2} = \sqrt{\overline{25^2 - \overline{\tfrac{151}{10}}\Big|^2} \times 2} = \frac{1}{10}\sqrt{\overline{250^2 - 151^2} \times 2} = \frac{1}{10}\sqrt{16898}$; and, by similar triangles, $An + mD : mn :: An : nr = \frac{420\sqrt{2}}{175 + \sqrt{8449}}$; hence $rA = \sqrt{An^2 + nr^2} = \frac{\sqrt{\frac{1}{2} \times 35^2 \times \overline{175 + \sqrt{8449}}\big|^2 + 2 \times 420^2}}{175 + \sqrt{58449}}$; and, by similar triangles, $rn : nm :: rA : AD = \sqrt{793 + 7\sqrt{8449}} = BE$.—Then, becauſe AD is equal and perpendicular to BE, by rule 2 page 69 we obtain the area $= AD \times BE \times \frac{1}{2}$ ſine $LF = \frac{1}{2} BE^2 = \frac{1}{2} \times \overline{793 + 7\sqrt{8449}} = 718{\cdot}214547$ ſquare poles $= 4$ac. 1ro. $38{\cdot}214547$ perches.

Otherwiſe.

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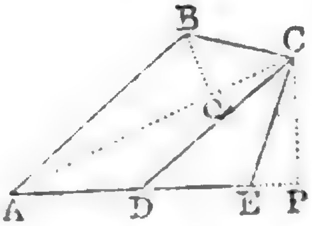
[*] For, by the conſtruction, AB, BD are the two ſides 35, 25, and the diagonal $BE = AD$; ſo that we have only to ſhew that AE, ED are equal to the other two ſides, 31, 19.—Now, by the conſtruction, the angles CAD, ABE, being each the complement of BAD, are equal to each other; and, by the conſtruction, CA and AD, AB and BE, about thoſe equal angles, are alſo equal; and therefore the remaining ſides CD, AE are, likewiſe, equal, that is $AE = 31$.—But, by the property of all △s, $BA^2 - AE^2 = BF^2 - FE^2 = BD^2 - DE^2$; and, therefore, $DE = \sqrt{BD^2 + AE^2 - AB^2} = \sqrt{25^2 + 31^2 - 35^2} = 19$. *Q.E.D.*

Otherwiſe.

Having found $mB = \frac{151}{10}\sqrt{2}$; we ſhall have as $DB = 25 : Bm = \frac{151}{10}\sqrt{2} :: 1 : \frac{151}{250}\sqrt{2} = {\cdot}604\sqrt{2} = {\cdot}85418499$ = the coſine of the $\angle DBm = 31^\circ\ 20'$; but $mBA = 45^\circ$, conſequently $\angle DBA = 76^\circ\ 20'$; then $60 = AB + BD : 10 = AB - BD ::$ tang. $\frac{\angle BDA + \angle BAD}{2} = 51^\circ\ 20'$: tang. $\frac{\angle BDA - \angle BAD}{2} = 11^\circ\ 58'$, hence $51^\circ\ 50' - 11^\circ\ 58' = 39^\circ\ 52' = \angle BAD$; laſtly, s. $\angle BAD$: s. $\angle DBA :: BD = 25 : DA = 37{\cdot}897 = BE$, which is nearly equal to $\sqrt{793 + 7\sqrt{8449}} = 37{\cdot}90025$, the value above found.

QUESTION XXI.

The trapezium *ABCD* is to be meaſured; and, becauſe of ſome obſtructions from wood, water, &c. or for want of proper inſtruments, there can be taken only the following meaſures: viz. *AB* is 58·5 chains, *BC* 27·3, *CD* 50, and *DA* 32 chains; alſo in *AD*, continued, there is meaſured any diſtance *DE* 27·5, and the diſtance from *E* to *C* is 32·5 chains; required the area.



Draw the diagonal *AC*, and on *AD*, produced, let fall the perpendicular *CP*.

Becauſe DC^2 is $= CE^2 + ED^2 + 2DE \times EP$, we ſhall have $EP = \frac{DC^2 - CE^2 - ED^2}{2DE} = \frac{50^2 - 32{\cdot}5^2 - 27{\cdot}5^2}{55} = 12{\cdot}5$; but $PC = \sqrt{CE^2 - FP^2} = \sqrt{32{\cdot}5^2 - 12{\cdot}5^2} = 30$, therefore $\frac{AD \times PC}{2} = 16 \times 30 = 480 =$ the area of the triangle *ADC*.

Again, $AP = AD + DE + EP = 32 + 27{\cdot}5 + 12{\cdot}5 = 72$; therefore $CA = \sqrt{AP^2 + PC^2} = \sqrt{72^2 + 30^2} = 78$.

Alſo, letting fall the perpendicular BQ, ſince $AB^2 = BC^2 + CA^2 - 2AC \times CQ$, we ſhall have $CQ = \frac{BC^2 + CA^2 - AB^2}{2AC} = 21{\cdot}84$; hence $QB = \sqrt{BC^2 - CQ^2} = 16{\cdot}38$; therefore $\frac{AC \times QB}{2} = 39 \times 16{\cdot}38 = 638{\cdot}82 =$ the triangle ABC.

Whence $480 + 638{\cdot}82 = 1118{\cdot}82$ ſquare chains $=$ $111{\cdot}882$ acres $=$ 111 ac. 3 r. $21{\cdot}12$ perches $=$ the whole area required.

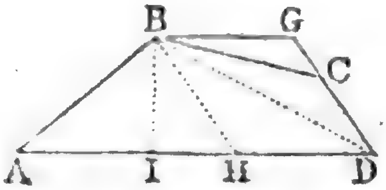
QUESTION XXII.

To find the length of a chord cutting off any part, as ſuppoſe $\frac{1}{3}$, from a circle whoſe diameter is 289.

The area of the whole tabular circle, or of the circle whoſe diameter is 1, is $\cdot78539816$; the $\frac{1}{3}$ of which is $\cdot26179939$ nearly, which is a ſegment ſimilar to the ſegment to be cut off. Now, the verſed ſine anſwering to this area, in the table of circular ſegments, is $\cdot36753395$, which taken from the whole diameter 1 leaves $\cdot63246605$ for the other part of the diameter; then, becauſe the ſemi-chord is a mean proportional between the two ſegments of the diameter, we ſhall have $2\sqrt{\cdot63246605 \times \cdot36753395} = \cdot96426162 =$ the chord of the tabular ſimilar ſegment. Wherefore as $1 : 289 :: \cdot96426162 : 278{\cdot}6716 =$ the chord line required.

QUESTION XXIII.

In the unpaſſable field $ABCD$ there are meaſured the ſides AB, 29; AD, 59; and DC, 15 chains: alſo, by drawing BG parallel to AD till it meet with DC, produced, in G, BG meaſures 23; and GC, 10 chains; required the content.



Draw BH parallel to GD: then is $BHDG$ a parallelogram, and conſequently $BH = GD = 25$, and $HD = BG = 23$; hence $AH = AD - DH = 36$.

But, letting fall the perpendicular BI, $HI = \frac{BH^2 + HA^2 - AB^2}{2AH} = 15$, and $IB = \sqrt{BH^2 - HI^2} = 20$. Whence $\overline{AD+BG} \times \frac{1}{2} IB = 82 \times 10 = 820 =$ the area of the trapezoid $ABGD$.

Again, drawing the diagonal BD, $HD \times \frac{1}{2} IB = 23 \times 10 = 230 =$ the triangle $BHD =$ the triangle BGD; but the triangles BGD, BGC, being of the ſame height, are to each other as their baſes DG, GC; that is, $DG = 25 : GC = 10 ::$ triangle $BDG = 230$: triangle $BCG = \frac{2}{5} \times 230 = 2 \times 46 = 92$.

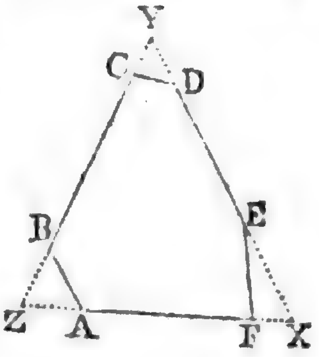
Whence, by taking the triangle BCG from the trapezoid $ABGD$, we have $820 - 92 = 728$ ſquare chains $= 72$ a. 3 r. 8 perches $=$ the area required.

Note. It is evident that this figure will be conſtructed, by firſt forming the triangle ABH, and joining the parallelogram $BGDH$ to it.

QUESTION XXIV.

To find the area of the hexagonal field $ABCDEF$, the ſides BC, DE, FA, being produced to meet in the points Z, Y, X; there are given the following dimenſions; $ZB = 39$, $BC = 118$, $CY = 25$, $YD = 30$, $DE = 100$, $EX = 65$, $XF = 23$, $FA = 109$, and $AZ = 37$ chains.

Firſt,



Firſt, by prob. 2 ſect. 1, the area of the triangle ZYX is 14196 ſquare chains.

But, it appears from rule 3 of the ſame problem that, the areas of triangles, having one angle common, are as the rectangles of the ſines including the common angle; wherefore

$XZ \times ZY = 169 \times 182 : 37 \times 39 :: 14196 : 666 = \triangle AZB$,

$ZY \times YX = 182 \times 195 : 25 \times 30 :: 14196 : 300 = \triangle CYD$,

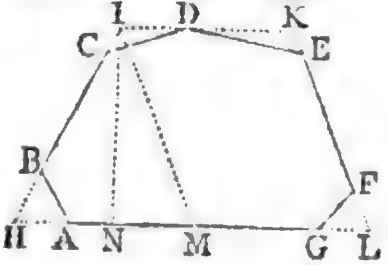
$YX \times XZ = 195 \times 169 : 65 \times 23 :: 14196 : \underline{644} = \triangle EXF$,

whoſe ſum 1610 taken from 14196 the triangle ZYX, leaves 12586 = the area of the hexagon $ABCDEF$ required.

QUESTION XXV.

It is required to find the area of the heptagon $ABCDEFG$ inſcribed in the trapezoid $HIKL$, the dimenſions as below:

Viz. $HB=34$, $BC=85$, $CI=17$, $ID=42$, $DK=66$, $KE=$, $EF=91$, $FL=26$, $LG=29$, $GA=160$, and $AH=33$.



Firſt, drawing IM parallel to KL, and letting fall the perpendicular IN, IM will be $= KL = 130$, $MH = HL - IK = 114$; hence $MN = \frac{IM^2 + MH^2 - HI^2}{2MH} = 50$, and $NI = \sqrt{IM^2 - MN^2} = 120$.

But a trapezoid is to a triangle, having each one angle equal, as the rectangle under the ſum of the parallel ſides and the other ſide including that angle, in the trapezoid, is to the rectangle under the ſides including the angle, in the triangle.

Or, As that one of the two ſides in the trapezoid, about the angle, which is not one of the parallel ſides, is to half the diſtance of the parallel ſides, ſo is the rectangle under the two ſides of the triangle about the angle, to its area.*

$$\text{Wherefore}\begin{cases}136:60::33\times34:495=\triangle AHB,\\ 136:60::17\times42:315=\triangle CID,\\ 130:60::66\times13:396=\triangle DKE,\\ 130:60::26\times29:348=\triangle FLG,\end{cases}$$

the ſum is 1554, which taken from 19800, the area of the trapezoid, leaves 18246 ſquare chains = 1824 a. 2 r. 16 perches.

Note. This is conſtructed as the 23d.

QUESTION XXVI.

In the quadrangular field *ABCD*, if *AD* be = 46·8, *DC* = 19·5, *BE* perpendicular to *AD* and = 44·8 chains, and *D* a right angle; alſo if *AB* be to *BC* as 8 to 5; it is required to find the content of the trapezium *ABCD*, and the lengths of the ſides *AB*, *BC*.

Conſtruction.

Draw *AD* = 46·8, and *DC* = 19·5 making a right angle *D* with it; join *A*, *C*; take *AF* to *AC* as 8 to 13 (8+5)

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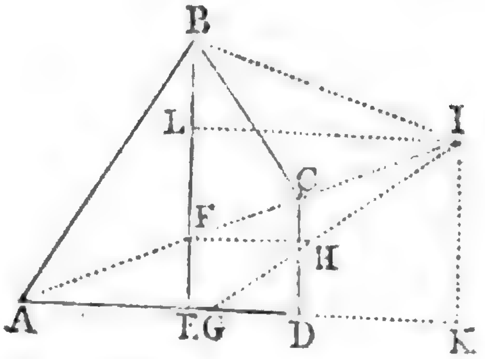
* For $AH\times HB : \triangle BAH :: (\text{radius} : \tfrac{1}{2}\text{s}.LH ::)\ HI : \tfrac{1}{2}IN ::$ (by equal mult.) $HI\times\overline{IK+HL} : \tfrac{1}{2}IN\times\overline{IK+HL} =$ the trapezium.

In the ſame manner, $GL\times LF : \triangle GLF :: LK : \tfrac{1}{2}IN :: LK\times\overline{IK+HL} : \tfrac{1}{2}IN\times\overline{IK+HL}$.

But the angles *I* and *K* are the ſupplements of the angles *H* and *L*, and have, therefore, the ſame ſines; and, conſequently, the rule will be the ſame for them.



(8+5), or $AF:FC::8:5$; make $AG=AF$, and FH parallel to it and = to FC; draw GHI meeting AC, produced, in I; with the center I and radius IF, deſcribe an arc meeting a line drawn parallel to and at the diſtance of 44·8 (the perpendicular) from AD, in B the other point of the figure required.*

Calculation.

Firſt, $AC=\sqrt{AD^2+DC^2}=50{\cdot}7$, and, by the conſtruction, $13:8::50{\cdot}7=AC:31{\cdot}2=AF$; hence $FC=CA-AF=19{\cdot}5$; then, by ſimilar triangles, $11{\cdot}7\ (=AG-FH=AF-FC):31{\cdot}2\ (=AF=AI-IF)::19{\cdot}5\ (=FH=FC):52=FI=IB$; whence $AI=AF+FI=83{\cdot}2$; then, by ſimilar triangles again, (demitting IK perpendicular to AD, produced) $50{\cdot}7=AC:83{\cdot}2=AI::\left\{\begin{array}{l}46{\cdot}8=AD:76{\cdot}8=AK,\\ 19{\cdot}5=DC:32\quad=KI,\end{array}\right\}$ whence (drawing IL parallel to KE) $LB=BE-KI=12{\cdot}8$, and $KE=LI=\sqrt{IB^2-BL^2}=50{\cdot}4$; and hence $EA=AK-KE=26{\cdot}4$, and $AB=\sqrt{BE^2+EA^2}=52$; alſo $8:5::52=AB:32{\cdot}5=BC$.

But $\frac{1}{2}\ AE\times EB=13{\cdot}2\times44{\cdot}8=591{\cdot}36=$ the area of the triangle ABC.

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* DEMONSTRATION.

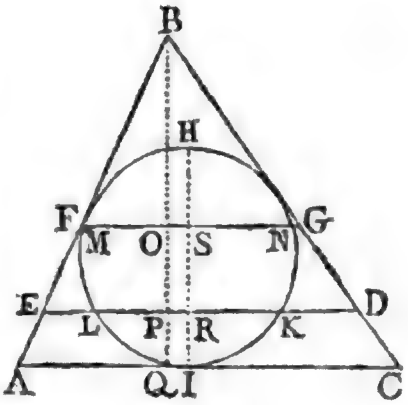
Becauſe, by ſimilar Δs, $IA:AG\ (AF)::IF:FH\ (FC)$, we have, by diviſion, $AI:IF::FI:IC$, or, joining I, B, $AI:IB::BI:IC$; but the $\angle I$, included by theſe proportional lines, is common to both the Δs ABI, IBC, $\because$ thoſe Δs are ſimilar, and conſequently $AB:BC::(AI:IB\ (IF)::AG:FH::)\ 8:5$.

And $\overline{EB+CD}\times\frac{1}{2}DE = 64{\cdot}3\times 10{\cdot}2 = 655{\cdot}86 =$ the area of the trapezoid *DEBC*.

Therefore $655{\cdot}86 + 591{\cdot}36 = 1247{\cdot}22$ ſquare chains $=$ 124a. 2r. 35·52 perches $=$ the area of the quadrangle *ABCD* required.

QUESTION XXVII.

Given the three ſides $AB = 13$, $BC = 15$; and $CA = 14$, of a triangle *ABC*, divided into three equal parts by the two lines *FG*, *ED* both parallel to *AC*; it is required to find the areas of the two ſegments *MHN*, *KIL*, and zone *LMNK* into which the inſcribed circle is cut by thoſe lines.



Firſt, $\sqrt{21\times8\times7\times6} = \sqrt{7\times3\times2\times4\times7\times2\times3} = 7\times3\times4 = 84 =$ the area of the triangle *ABC*.

Then $84\div7\,(\frac{1}{2}AC) = 12 =$ the perpendicular BQ; and $84\div21\left(\frac{AB+BC+CA}{2}\right) = 4 =$ the radius, or $8 =$ the diameter *HI* of the inſcribed circle.

Now, the triangles *BFG*, *BED*, *BAC*, are ſimilar; but ſimilar triangles are as the ſquares of their like dimenſions; alſo thoſe triangles are to one another as the numbers 1, 2, 3, reſpectively; wherefore $\sqrt{3} : 12 = BQ :: \begin{cases}\sqrt{1} : 12\sqrt{\frac{1}{3}} = 4\sqrt{3} = BO, \\ \sqrt{2} : 12\sqrt{\frac{2}{3}} = 4\sqrt{6} = BP;\end{cases}$

Hence $QB - BP = 12 - 4\sqrt{6} = PQ =$ the verſed ſine IR; $QB - BO = 12 - 4\sqrt{3} = OQ = IS$; and $HI - IS = 4\sqrt{3} - 4 =$ the verſed ſine SH.

Then, dividing the verſed ſines *HS*, *RI*, by the diameter, we have $\frac{4\sqrt{3}-4}{8} = \frac{\sqrt{3}-1}{2} = \cdot366025404$, and $\frac{12-4\sqrt{6}}{8} = \frac{3-\sqrt{6}}{2} = \cdot275255128$; the correſponding tabular verſed ſines; to which, in the table of circular ſegments, belong the areas ·26034449, and ·17577983 = the tabular area correſponding to the two ſegments; whoſe ſum taken from ·785398 &c. leaves ·34927384 = that correſponding to the zone; each of which being multiplied into 64, the ſquare of the diameter, gives their ſeveral areas, viz.

$$64\times\begin{cases}\cdot26034449 = 16\cdot66204736 = \text{the ſegment } LIK,\\ \cdot17577983 = 11\cdot24990912 = \text{the ſegment } NHM,\\ \cdot34927384 = 22\cdot35352576 = \text{the zone } MLKN.\end{cases}$$

PART III.

MENSURATION *of* SOLIDS.

GENERAL DEFINITIONS.

A Solid or body is a figure extended in every direction. It is commonly ſaid to conſiſt of length, breadth, and thickneſs, which are three of its extentions; of which the direction of each is perpendicular to thoſe of the other two.

The meaſure of a ſolid is called its ſolidity, capacity, or content.

By the menſuration of ſolids then are determined the ſpaces included by contiguous ſurfaces; and the ſum of the meaſures of theſe including ſurfaces is the ſurface or ſuperficies of the body.

Solids are meaſured by cubes, whoſe ſides are inches, feet, yards, or any other aſſigned quantity; and hence the ſolidity of a body is ſaid to be ſo many cubic inches, feet, yards, &c. as will fill its capacity or ſpace, or another of an equal magnitude.

The leaſt ſolid meaſure is the cubic inch, other cubes being taken from it according to the proportion in the following

TABLE.

Digitized by

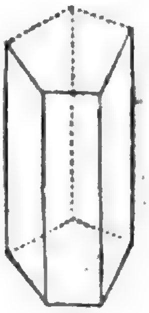
Table *of* Solid Measure.

Cubic inches	cubic feet				
1728	1	cubic yards			
46656	27	1	cub. poles		
7762392	4492$\frac{1}{8}$	166$\frac{3}{8}$	1	c. furl.	
946793088000	28749600	10648000	64000	1	c mile
254358061056000	147197952000	5451776000	32768000	512	1

SECTION I.

Of Priſms, Pyramids, and the Sphere, with the Parts into which ſome of them may be cut by planes.

A Priſm is a ſolid all whoſe plane ſections, parallel to its end or baſes, are equal and ſimilar figures.—It is among ſolids what a parallelogram is among planes.



Or a priſm is a ſolid whoſe ſides are parallelograms, the ends of all which parallelograms form the baſes or ends of the priſm, which ends or baſes are two equal, parallel, and ſimilar plane figures.

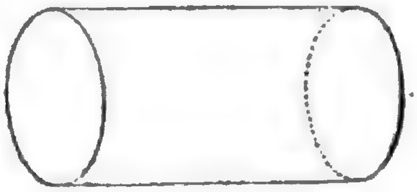
Priſms receive ſeveral particular denominations according to the figure of their ends or baſes.

If the baſe be a triangle, it is called a triangular priſm. And is ſomething like a hat box.

If its baſe be a ſquare, it is a ſquare priſm; if a pentagon, a pentagonal priſm; if a hexagon, a hexagonal priſm; &c.

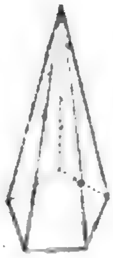
If the length or height of a ſquare priſm be equal to one ſide of its baſe, it will be of the form of a die, and is called a cube.

If the baſe be a rectangle, it is called a parallelopiped: like a box or cheſt whoſe length and breadth or height are unequal.



If the baſe of a priſm be a circle, it is called a cylinder: like a rowling-ſtone.

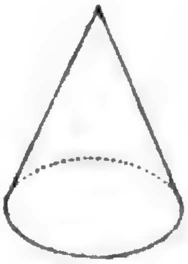
In this priſm the number of parallelograms forming its ſides is infinite.



A pyramid is a ſolid whoſe ſides are triangles, and whoſe vertexes meet in a point at the top called the vertex of the pyramid. Alſo the baſes of the triangles form the baſe of the pyramid.

If a right line, moveable about a point fixed above a plane, be moved round ſo as to deſcribe any figure upon the plane, the ſpace incloſed by the path of the line will be in the form of a pyramid.

The pyramid, like the priſm, receives particular denominations according to the figure of its baſe. Thus a triangular, ſquare, pentagonal, hexagonal, &c. pyramid hath for its baſe a triangle, ſquare, pentagon, hexagon, &c.



A cone is a round pyramid, or a pyramid having a circular baſe.

Here the number of triangles forming the upright ſurface of the pyramid is infinite.

Hence, if a ſector of a circle be conceived to be rowled up till its two bounding radiuſes coincide, it will take the form of a cone. And, in general, a pyramid may be formed by rolling up a part of a plane figure, cut off from it by two lines drawn to the perimeter from any point within it; having cut the figure through from the ſaid point to every angular point of the perimeter, to make it bend or join ſo as that every part into which it is cut may form a right-lined angle with the next part.

A ſphere is a ſolid bounded by a convex ſurface, every point of which is equally diſtant from a point within called the center.

If a ſemicircle be moved round its diameter fixed, the ſpace through which it moves will be in form of a ſphere or globe.

The axe or axis of a ſolid is a line connecting the middle of its oppoſite ends. So the axe of a priſm is a line from the middle of the one end to the middle of the other; the axe of a pyramid is a line from the vertex to the middle of the baſe; and the axe or diameter of a ſphere is a line drawn through the center and terminated at the ſurface on both ſides.

A right priſm, or pyramid is a priſm or pyramid whoſe axe is perpendicular to its baſe. And

An oblique priſm or pyramid is one whoſe axe is not perpendicular to its baſe.

Alſo, a priſm, or a pyramid, is ſaid to be regular or irregular according as the baſe is a regular or an irregular plane figure.

A ſegment of a pyramid, ſphere, or any other tapering ſolid, is a part cut off the top by a plane parallel to the baſe.—If the plane paſs through the center of the ſphere, it will cut it equally in two, and each half is called a hemiſphere.

The ſection made by the plane is a plane ſimilar to the baſe of the figure; and every ſection of a ſphere is a circle. If the ſection be made through the center, it is a great circle of the ſphere, having the ſame diameter with the ſphere; if not, it is a little circle.

A fruſtum, truncus, or trunk, is the part remaining at the bottom after the ſegment is cut off.

If a fruſtum be cut by a plane diagonally paſſing through the extremity of one ſide at the leſs end, and through the extremity of the oppoſite ſide at the greater end, the two parts into which it is cut are called ungulas or hoofs; the greater hoof being that including the greater end, and the leſs that including the leſs.

A zone of a ſphere is a part intercepted between two parallel planes.

The middle zone is that whoſe parallel ends are equally diſtant from the center.

A ſector of a ſphere is a ſolid compoſed of a ſegment leſs than a hemiſphere, and of a cone having the ſame baſe with the ſegment, and whoſe vertex is the center of the ſphere.

The height of a ſolid is the perpendicular drawn from its vertex or top to the plane of its baſe.

A ſolid is convex where the ſurface, being curved, riſes outwards; and concave where it ſinks inwards.

Similar pyramids are thoſe which are contained under an equal number of ſimilar, and ſimilarly poſited plane figures.

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Similar ſolids are thoſe which are made up of an equal number of ſimilar, and ſimilarly poſited pyramids; or that are contained under an equal number of ſimilar plane figures, alike placed.

PROBLEM I.

To find the ſurface of a priſm.

General RULE.

It is evident, that, if the area of each ſide and end be calculated ſeparately, the ſum of thoſe areas will be the whole ſurface of any priſm, whether right or oblique, or, indeed, of any other body whatever.[*] —But for a right priſm obſerve the following

Particular RULE.

Multiply the perimeter of the end into the height, and the product will be the ſum of the ſides, or upright ſurface. Or, if the ends of the priſm be regular plane figures, multiply the perimeter of the end into the ſum of the height of the priſm and the radius of the circle inſcribed in the end, and the product will be the whole ſurface.

EXAMPLE I. What is the upright ſurface of a triangular priſm whoſe length is 20 feet, and the ends of its baſe each 18 inches?

Here $18\times3=54$ inches $=4\frac{1}{2}$ feet equal to the perimeter of the baſe.

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[*] The ſurfaces of ſimilar priſms, and indeed of any other ſimilar bodies, are as the ſquares of their like lineal dimenſions; this follows from their being compoſed of ſimilar plane figures, alike placed.

Therefore $4\frac{1}{2}\times20=90$ ſquare feet equal to the upright ſurface.

Again, by rule 2 problem 4 ſection 1, we have $2\times\frac{3}{2}\times\frac{3}{2}\times\cdot433013=1\cdot9485585$ = the area of the two ends.

Whence $91\cdot9485585$ is the whole ſurface.

EXAMPLE II. What is the ſurface of a cube the length of each of whoſe ſides is 20 feet?

$20\times20=400$ equal to the area of one ſide,

And $400\times6=2400$ = the whole ſurface required.

EXAMPLE III. What muſt be paid for lining a rectangular ciſtern with lead at 2d. a lb. the lead being 7 feet to the lb. ſuppoſing (on the inſide) the length to be 3 feet 2 inches, the breadth 2 feet 8 inches, and height 2 feet 6 inches?

Firſt, $\overline{38+32}\times2\times30=70\times60=4200$ ſquare inches equal to the two ſides and two ends together.

Then, $38\times32=1216$ = to the area of the bottom.

Wherefore $4200+1216=5416$ ſquare inches = $37\frac{11}{18}$ ſquare feet = the whole area.

But $37\frac{11}{18}\times7=263\frac{5}{18}$ lb. = the whole weight.

Therefore 1 lb. : 2d. :: $263\frac{5}{18}$: 2l. 3s. $10\frac{5}{9}$d. the coſt.

EXAMPLE IV. What is the convex ſurface of a round priſm or cylinder whoſe length is 20 feet, and the diameter of whoſe baſe is 2 feet?

$3\cdot1416\times2=6\cdot2832$ = the circumference.

Therefore $6\cdot2832\times20=125\cdot664$ = the convex ſurface.

EXAMPLE V. What is the whole ſurface of a cylinder whoſe length is 10 feet, and the circumference is 3 feet?

Here $\overline{10+\frac{3}{2\times3\cdot1416}}\times3=30+\frac{9}{6\cdot2832}=31\cdot43239$ = the whole ſurface required.

PROBLEM II.

To find the ſolidity of a priſm.

RULE.

Multiply the area of the baſe into the height, and the product will be the ſolidity.*

EXAMPLE I.

How many ſolid feet are in a ſquare priſm whoſe length is $5\frac{1}{2}$ feet, and each ſide of its baſe is $1\frac{1}{3}$ feet?

$1\frac{1}{3}\times1\frac{1}{3}=\frac{4}{3}\times\frac{4}{3}=\frac{16}{9}$ = the area of the end. And $\frac{16}{9}\times5\frac{1}{2}=\frac{16}{9}\times\frac{11}{2}=\frac{88}{9}=9\frac{7}{9}$ ſolid feet equal to the content required.

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* For, if we conceive to be cut off from the priſm, by a plane parallel to the ends, a part whoſe height is equal to the lineal meaſuring unit; and then imagine both its ends to be divided in a ſimilar manner into ſo many ſquares as are expreſſed by the area of each, the ſide of each of thoſe ſquares being equal to the lineal meaſuring unit; and, laſtly, ſuppoſe planes to be drawn through the correſponding lines of diviſion; it will be evident that the part cut off will be divided, by thoſe planes, into ſo many cubes as there are ſquares in each end, and alſo having the ſame dimenſion with thoſe ſquares, viz. the lineal meaſuring unit; and this number of cubes is the meaſure of the part.

But the magnitude of the whole priſm, or, indeed, of any other of an equal baſe, is, evidently, to the magnitude of the part whoſe height is the lineal meaſuring unit, as the length of the whole is to that unit (1).

And, therefore, the meaſure of the whole is equal to that of the part ſo often repeated as there are lineal meaſuring units in the height; that is, equal to the baſe drawn into the height.

And that the rule is true for oblique priſms as well as right ones, will be evident by conceiving, according to the method of *Cavalerius*, a right and an oblique priſm, of equal baſes and heights, to be made up of an indefinite number of equal thin plates all parallel to the baſe; for the priſms being both of a height, the one of them will require as many of ſuch plates to compoſe it as the other.

EXAMPLE II.

How many ale gallons of water will the ciſtern hold whoſe length, breadth, and height are 3 feet 2 inches, 2 feet 8 inches, and 2 feet 6 inches?

$38 \times 32 \times 30 = 36480 =$ the ſolidity in inches. And $36480 \div 282 = 6080 \div 47 = 129\frac{17}{47}$ ale gallons.

EXAMPLE III.

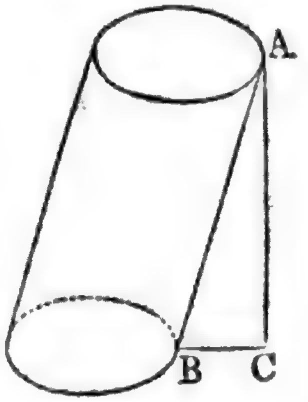
What is the capacity of a cylinder whoſe height is 20 feet, and the circumference of its baſe 20 feet, alſo?

Firſt, $\frac{20}{3{\cdot}1416} =$ the diameter, and $10 \times \frac{10}{3{\cdot}1416} = \frac{100}{3{\cdot}1416} = \frac{25}{{\cdot}7854} =$ the area of the end.

Then, $\frac{25}{{\cdot}7854} \times 20 = \frac{2500000}{3927} = 636{\cdot}61828 =$ the capacity required.

EXAMPLE IV.

What is the capacity of the oblique cylinder whoſe axe is 20 feet, and the circumference of whoſe baſe is 20 feet alſo; the axe making an angle of 75° with the baſe?



As in the laſt example, the area of the baſe is $\frac{25}{{\cdot}7854}$ or $\frac{125000}{3927}$.

But as 1 (rad.) $: {\cdot}9659258$ (s. $75° = \angle B$) $:: 20$ (AB) $: {\cdot}9659258 \times 20 = 19{\cdot}318516 = AC =$ the height of the cylinder.

Therefore $\frac{125000}{3927} \times 19{\cdot}318516 = 125000 \times {\cdot}0049194082 = 614{\cdot}92602$ the capacity required.

Other-

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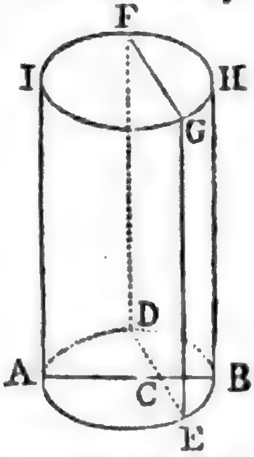
Otherwiſe.

As 1 (s. ∠*C*) : ·9659258 (s. ∠*B*) :: 636·61828 (the capacity in the laſt example) : 636·61828 × ·9659258 = 614·92602 the capacity the ſame as before.

EXAMPLE V.

Suppoſe the right cylinder whoſe length is 20 feet, and diameter 3 feet, to be cut by a plane parallel to and at the diſtance of 1 foot from its axe: required the ſolidities of the two priſms into which the cylinder is cut.

$\frac{CB}{BA} = \frac{\frac{1}{2}}{3} = \frac{1}{6} = \cdot 1\dot{6}$ = the tab. verſed ſine, to which in the table of circular ſegments correſponds the area ·08604117; hence ·08604117×9 (AB^2) = ·77437053 = the area of the ſeg. *BDE*, and ·78539816 − ·08604117 × 9 = ·69935699 × 9 = 6·29421291 = the ſegment *ADE*.

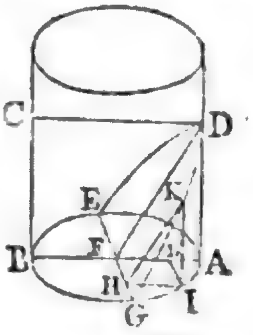


$$\text{Whence } 20 \times \begin{cases} \cdot 77437053 = 15\cdot 48741 = \text{ſlice } FGHBED \\ 6\cdot 29421699 = 125\cdot 88434 \quad\quad FIGEAD. \end{cases}$$

PROBLEM III.

To find the curve ſurface of the ungula DEACD *of a cylinder.*

RULE.



Put h = the height *AD*,
v = *AF* the verſ. ſine of ½ the arc of the baſe,
d = the diameter *AB*,
a = the arc *EAG* of the baſe,
s = *FG* the right ſine, and c = the coſine of the ½ arc;

Then $\frac{h}{v}\times\overline{ds-ac}=$ the convex ſurface.*

That is, From the product of the diameter and ſine, ſubtract the product of the arc and coſine, and multiply the difference by the quotient ariſing from the diviſion of the height by the verſed ſine.

Note 1. When F is the center of the baſe; then $v=s=\frac{1}{2}d$, and $c=0$; and then the theorem becomes dh, viz. the product of the diameter and height equal to the curve ſurface.

Note 2. When AF exceeds $\frac{1}{2}AB$, then ac muſt be added.

EXAMPLE I.

Given the diameter AB equal to 100, the height AD equal to 140, and the verſed ſine AF equal to 10: required the curve ſurface.

Here

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* For, having drawn HI, IK parallel to FA and AD reſpectively, and joined the points H, K, ſince it is evident that the ſurface is generated by the motion of IK along the arc AIG, $KI\times$ the fluxion of IA will be the flux. of the ſurface. Wherefore put $z=AI$, $x=$ its ſine $IL\|GE$, and $y=$ its coſine; then $HI=LF=FA-AL=v-\frac{1}{2}d+y$, and, by ſimilar triangles, $FA:AD::HI:IK=\frac{h}{v}\times\overline{v-\frac{1}{2}d+y}$; and hence the fluxion of the ſurface, or $\dot{z}\times IK$ is $\frac{h}{v}\times\overline{v\dot{z}-\frac{1}{2}d\dot{z}+y\dot{z}}$. But $y\dot{z}=\frac{1}{2}d\dot{x}$, and therefore $\dot{z}\times IK=\frac{h}{v}\times\overline{v\dot{z}-\frac{1}{2}d\dot{z}+\frac{1}{2}d\dot{x}}$; the fluent of which is $\frac{h}{v}\times\overline{vz-\frac{1}{2}dz+\frac{1}{2}dx}$ $=\frac{h}{v}\times\overline{\frac{1}{2}dx-cz}=$ (when $IA=AG$) $\frac{h}{v}\times\overline{\frac{1}{2}ds-\frac{1}{2}ac}$, the double of which is $\frac{h}{v}\times\overline{ds-ac}=$ the whole convex ſurface $DEAGD$.

COROLLARY I.

If F be the center; then $v=s=\frac{1}{2}d$, and $c=0$; and then the theorem becomes barely $dh=4$ times the triangle DAF.

COROLLARY II.

When AF exceeds $\frac{1}{2}d$, c is negative, and then $-ac$ becomes $+ac$.

COROLLARY III.

If F coincide with B; then $s=0$, and $c=-\frac{1}{2}v$; and the theorem becomes $\frac{1}{2}ah=$ half the ſurface of the cylinder.

Here $d=100$, $b=140$, and $v=10$: wherefore $\frac{1}{2}d-v=50-10=40=c$. And $\sqrt{\tfrac{1}{4}dd-cc}=\sqrt{2500-1600}=\sqrt{900}=30=s$.

But $\frac{s}{\frac{1}{2}d}=\frac{30}{50}=\frac{3}{5}=\cdot 6=$ the ſine reduced to the radius 1, to which, in a table of ſines, belong 36 degrees, 52·268 minutes $=36\cdot 87113$ degrees. Then, by rule 1 for the length of a circular arc, $\cdot 01745329\times 36\cdot 87113\times 100=64\cdot 352252=$ the arc a.

Whence $\frac{b}{v}\times\overline{ds-ac}=14\times\overline{3000-2574\cdot 09008}=14\times 425\cdot 90992=5962\cdot 73888=$ the convex ſurface required.

EXAMPLE II.

If the diameter and height be 100 and 140, as before, and the ſection be made through the center of the baſe, or $v=\frac{1}{2}d=50$; what is the convex ſurface?

Here, by note 1, $db=100\times 140=14000=$ the convex ſurface required.

EXAMPLE III.

Suppoſing d and b ſtill the ſame, and $v=90$; to find the convex ſurface.

Here $\frac{1}{2}d-v=50-90=-40=c$, $s=30$ the ſame as before, but is, here, the ſine of the ſupplemental arc, which therefore is $180-36\cdot 87113=143\cdot 12887$ degrees. Hence $\cdot 01745329\times 143\cdot 12887\times 100=249\cdot 807013=$ the arc a. Or the arc may be ſooner found by only ſubtracting the arc in the firſt example, viz. 64·352408, from 314·159 &c. the whole circumference.

Then, by note 2, $\frac{b}{v}\times\overline{ds+ac}=\frac{14}{5}\times\overline{3000+9992\cdot 28052}=\frac{14}{5}\times 12992\cdot 28052=20210\cdot 21414=$ the convex ſurface required.

PROBLEM IV.

To find the ſolidity of the hoof of a cylinder.

RULE.

From $\frac{2}{3}$ of the cube of the right ſine ſubtract the product of the baſe and coſine of half the arc of the baſe, then multiply the difference by the quotient of the height divided by the verſed ſine, and the product will be the ſolidity required.

That is, putting h = the height AD
v = the verſed ſine AF
s = the right ſine FG
c = the coſine $= \frac{1}{2}AB - AF$
} as before (ſee the laſt fig.)
and b = the baſe, or area of the ſeg. $GAEG$;

Then $\overline{\tfrac{2}{3}s^3 - bc} \times \frac{h}{v}$ = the ſolidity.*

Note

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* For the fluxion of the ſolid is = the $\triangle HIK$ drawn into the fluxion of LI, which fluxion will, therefore, be $\dot{x} \times \frac{h}{2v} \times HI^2$ (uſing the ſame characters as in the demonſtration of the laſt problem) $= \frac{h\dot{x}}{2v} \times \overline{y-c}|^2 = \frac{h\dot{x}}{2v} \times \overline{yy - 2cy + cc} = \frac{h\dot{x}}{2v} \times \overline{\tfrac{1}{4}dd - xx - 2cy + cc} = \frac{h\dot{x}}{2v} \times \overline{\tfrac{1}{4}dd - xx - dc + 2c \times AL + cc} = \frac{h\dot{x}}{2v} \times \overline{ss - xx - 2c \times FL}$; whoſe fluent $\left(\frac{hx}{2v} \times \overline{ss - \tfrac{1}{3}xx} - \frac{hc}{v} \times \text{area } FAIH\right)$ when I coincides with G, is $\frac{h}{2v} \times \overline{\tfrac{2}{3}s^3 - bc}$, the double of which is $\frac{h}{v} \times \overline{\tfrac{2}{3}s^3 - bc}$ = the content of the ſolid $DEAGD$ required.

COROLLARY I.

If F fall in the center of the baſe; then $c = 0$, and $s = v = \frac{1}{2}d$; and the rule will be $\frac{1}{6}ddh$.

COROLLARY II.

If AF exceed FB, c will be negative, and then $-bc$ will become $+bc$.

COROLLARY III.

If F fall in B; $s = 0$, and $c = -\frac{1}{2}v$; and then the theorem becomes $\frac{1}{2}bh$ = half the cylinder.

Note 1. If F be the center, that is, if the baſe be equal to the ſemicircle, then $v = s$, and $c = 0$; and therefore $\frac{2}{3}bss = \frac{1}{6}ddh$ is the ſolidity in that caſe.

Note 2. If v exceed $\frac{1}{2}d$, that is, if the baſe exceed the ſemicircle, then c is negative, and bc muſt be added.

EXAMPLE I.

If the diameter AB be 50, the height AD 120, and the verſed ſine AF 10; what is the ſolidity of the hoof.

Or ſuppoſing a cylindric veſſel $ABCD$, containing a fluid, to be placed in ſuch a poſition that the ſurface of the fluid, diſpoſing itſelf parallel to the horizon, may cut the baſe in GE leaving 40 inches of the diameter dry, and the ſide of the cylinder in D 120 inches diſtant from the baſe; to find how many ale gallons are in it; the diameter of the baſe being 50 inches.

Here $h = 120$, $d = 50$, and $v = 10$. Then $\frac{1}{2}d - v = 25 - 10 = 15 = c$, and $\sqrt{\frac{1}{4}dd - cc} = \sqrt{25^2 - 15^2} = \sqrt{5^2 \times \overline{5^2 - 3^2}} = \sqrt{5^2 \times 4^2} = 5 \times 4 = 20 = s$.

And, to find the baſe by the table of ſegments, $\frac{v}{d} = \frac{10}{50} = {\cdot}2$, this found in the column of verſed ſines, oppoſite to it I find the area ${\cdot}1118238$: hence $50 \times 50 \times {\cdot}1118238 = 279{\cdot}5595 = b =$ the ſegment or baſe.

Then $\frac{h}{v} \times \overline{\frac{2}{3}s^3 - bc} = 12 \times \overline{\frac{2}{3} \times 8000 - 15 \times 279{\cdot}5595} = 12 \times \overline{5333\frac{1}{3} - 4193{\cdot}3925} = 12 \times 1139{\cdot}9408 = 13679{\cdot}2896 =$ the ſolidity in inches; which divided by 282, the inches in a gallon, give $48{\cdot}50939$ ale gallons for the content.

EXAMPLE II.

Suppoſe the cylinder ſo placed, that the ſurface of the liquor may biſect the baſe, and riſe up the ſide to the ſame diſtance of 120 inches from the baſe: to find the content.

Here, by note 1, we have $\frac{1}{6}ddh = \frac{50\times50\times120}{6} = 50\times50\times20 = 50000$ ſolid inches $= 177{\cdot}3049645$ gallons, for the content in this caſe.

EXAMPLE III.

Suppoſe, now, the ſame veſſel ſo placed as that the ſurface of the liquor may leave only 10 inches of the diameter dry, ſtill riſing to the ſame diſtance of 120 inches along the ſide; to find the content.

Here the part of the cylinder's baſe left dry is equal to the baſe in the firſt example, viz. $279{\cdot}5595$; which, therefore, taken from $50\times50\times{\cdot}785398 16 = 1963{\cdot}4954$, the whole circle, leaves $1683{\cdot}9359 = b =$ the baſe of the ungula in this example.

Now $v=40$, $c=-15$, and $s=20$.

Whence $\frac{h}{v}\times\overline{\tfrac{2}{3}s^3-bc} = \frac{120}{40}\times\overline{\tfrac{2}{3}\times8000+25259{\cdot}0385} = 3\times30592{\cdot}3718 = 91777{\cdot}1154$ ſolid inches $= 325{\cdot}45076$ gallons $=$ the content in this caſe.

PROBLEM V.

To find the ſurface of a right pyramid.

RULE.

Multiply the perimeter of the baſe by the ſlant height or length of the ſide, and half the product will, evidently, be the ſurface, or the ſum of the areas of all the triangles which form it.*

Ex-

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* COROLLARY I.

Hence, becauſe in a right cone the circumference of the baſe into half the ſide is equal to the curve ſurface, and into half the radius is equal

EXAMPLE I.

What is the ſurface of a triangular pyramid, the ſlant height being 20 and each ſide of the baſe 3?

First, $3\times3=9=$ the perimeter of the baſe.

And $\frac{9\times20}{2}=9\times10=90=$ the ſurface required.

EXAMPLE II.

Required the ſurface of a ſquare pyramid whoſe ſlant height is 20 and each ſide of the baſe 3.

$\frac{3\times4\times20}{2}=12\times10=120=$ the ſurface ſought.

EXAMPLE III.

Required the convex ſurface of a circular pyramid, or cone, whoſe ſlant ſide is 20 and the circumference of the baſe 9.

$\frac{20\times9}{2}=10\times9=90=$ the convex ſurface.

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equal to the baſe, therefore as the radius of the baſe is to the ſide, ſo is the baſe to the curve ſurface.—And the ſame is true of any other pyramid whoſe baſe is a regular figure, viz. as the radius of the circle inſcribed in the baſe is to the ſlant height, ſo is the baſe to the ſurface.

COROLLARY II.

Let there be a cone and cylinder of the ſame baſe and altitude, the common altitude being equal to the radius of the baſe; then the baſe, the ſurface of the cone, and the ſurface of the cylinder, are to one another as the numbers 1, $\sqrt{2}$, and 2, and ſo are in continual proportion.

And the ſame is true of any other regular priſm and pyramid of the ſame baſe and altitude, the altitude being equal to the radius of the circle inſcribed in the baſe.

PROBLEM VI.

To find the ſurface of the fruſtum of a right pyramid.

RULE.

Multiply the ſum of the perimeters of the ends by the ſlant height, and half the product will be the ſurface required.*

EXAMPLE I.

How many ſquare feet are in the ſurface of the fruſtum of a ſquare pyramid whoſe ſlant height is 10 feet, each ſide of the greater baſe being 3 feet 4 inches, and each ſide of the leſs 2 feet 2 inches?

$4\times\overline{40+26}=4\times66=264=$ the ſum of the perimeters.

$\frac{264\times10\times12}{2}$ ſquare inches $=\frac{264\times10\times12}{2\times144}$ ſquare feet $=\frac{22\times10}{2}=11\times10=100$ feet $=$ the ſurface required.

EXAMPLE II.

If a ſegment of 6 feet ſlant height be cut off a cone whoſe ſlant height is 30 feet, and circumference of its baſe 10 feet, what will be the ſurface of the fruſtum?

As $30=$ the height of the whole : $6=$ the height of the upper part :: $10=$ the bottom circumference : $\frac{60}{30}=2=$ the circumference at the top of the fruſtum.

Then $\frac{10+2}{2}\times24=6\times24=144=$ the ſurface required.

PRO-

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* DEMONSTRATION.

For the ſurface is compoſed of a number of equal trapezoids whoſe common height is equal to the ſlant height of the fruſtum, and the ſums of whoſe parallel ſides make up the perimeters of the ends of the fruſtum.

PROBLEM VII.

To find the ſolidity of a pyramid.

RULE.

Multiply the baſe by the perpendicular height, and $\frac{1}{3}$ of the product will be the content.*

Ex-

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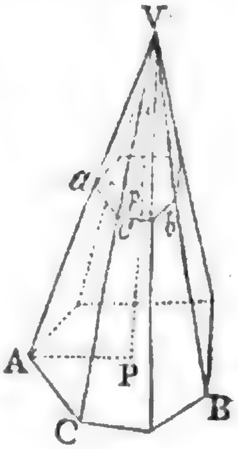
* Let VAB be a pyramid whoſe baſe is any plane figure, regular or irregular, VP a $\perp$ to the baſe in P and meeting the ſection ab, parallel to the baſe, in p; and put A for the area of the baſe AB, a for that of the ſection ab, h the height PV of the pyramid, and x the height Pp of the fruſtum $AabB$.



Firſt, the parallel ſections AB, ab, are to each other as the ſquares of their diſtances, PV, Vp, from the vertex. For, becauſe they are ſimilar plane figures, which are as the ſquares of their like ſides, we ſhall have $A : a :: AC^2 : ac^2 ::$ (by ſim. △s) $AV^2 : Va^2 ::$ (by ſim. △s again) PV^2 (hh) : Vp^2 $(\overline{h-x}|^2)$.

Hence we find $a = \overline{h-x}|^2 \times \frac{A}{hh}$, and $a\dot{x} = \frac{A\dot{x}}{hh} \times \overline{h-x}|^2$, the fluent of which is $Ax \times \overline{1 - \frac{x}{h} + \frac{xx}{3hh}} =$ the content of the fruſtum $AabB$.

COROLLARY I.

Since $\sqrt{A} : \sqrt{a} :: h : h-x$, it will be $\sqrt{A} - \sqrt{a} : \sqrt{A} :: x\,(h - \overline{h-x}) : h = \frac{x\sqrt{A}}{\sqrt{A}-\sqrt{a}}$, which being ſubſtituted for h in the Quantity $Ax \times \overline{1 - \frac{x}{h} + \frac{xx}{3hh}}$, expreſſing the content of the fruſtum, that content will become $\frac{1}{3}x \times \overline{A + a + \sqrt{Aa}}$.

COROLLARY II.

Or, if S be a ſide or any other line in the greater end, and s a ſimilar ſide or line in the leſs, ſince $S-s : S :: x : h$, hence $\frac{x}{h} = \frac{S-s}{S}$, which being ſubſtituted inſtead of it, gives $\frac{1}{3}Ax \times \frac{SS+Ss+ss}{SS}$ for the value of the ſaid fruſtum.

Co-

EXAMPLE I.

Required the ſolidity of a triangular pyramid whoſe height is 30 and each ſide of the baſe 3.

Firſt, $3 \times 3 \times {\cdot}433013 = 3{\cdot}897117$ = the area of the baſe.

Then $\frac{3{\cdot}897117 \times 30}{3} = 38{\cdot}97117$ = the ſolidity.

Ex-

COROLLARY III.

If $x = h$, then $a = 0$, and the general expreſſion becomes $\frac{1}{3} Ah =$ the whole pyramid; which is our rule.

COROLLARY IV.

If the baſe be a regular figure; let S be one of its ſides, $s =$ a ſide of the leſs end of the fruſtum, and $n =$ the area of a ſimilar figure whoſe ſide is 1, to be found in the table in page 81; then $A = nSS$, and $a = nss$; hence the ſolidity of the fruſtum will be $\overline{SS + Ss + ss} \times \frac{1}{3} nx = \frac{S^3 - s^3}{S - s} \times \frac{1}{3} nx$, and that of the pyramid $= \frac{1}{3} nSSh$.

COROLLARY V.

If the baſe be a ſquare; then $n = 1$, and the laſt expreſſions become $\frac{1}{3} SSh$ for the whole pyramid, and $\frac{1}{3} x \times \overline{SS + Ss + ss}$ or $\frac{S^3 - s^3}{S - s} \times \frac{1}{3} x$ for the fruſtum.

COROLLARY VI.

If the baſe be a circle; then $n = {\cdot}7854$, ${\cdot}2618 SSh =$ the cone, and $\overline{SS + Ss + ss} \times {\cdot}2618 x$ or $\frac{S^3 - s^3}{S - s} \times {\cdot}2618 x =$ the fruſtum; where S is the diameter of the baſe, and s that of the top.

COROLLARY VII.

If $a = A$; then $\overline{A + a + \sqrt{Aa}} \times \frac{1}{3} x$ becomes $3A \times \frac{1}{3} x = Ax =$ the priſm.

COROLLARY VIII.

Hence a priſm is to a pyramid of the ſame baſe and height as 3 to 1, and to the fruſtum of the ſame baſe and height as 1 to $1 - \frac{x}{h} + \frac{xx}{3hh}$, or as 3 to $1 + \frac{a}{A} + \sqrt{\frac{a}{A}}$, or as 3 to $\frac{SS + Ss + ss}{SS}$ or $1 + \frac{s}{S} + \frac{ss}{SS}$.

COROLLARY IX.

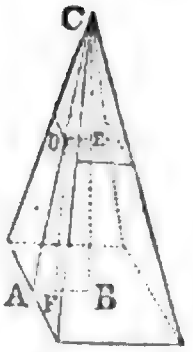
Similar pyramids are as the cubes of their like ſides. For the pyramid is as Ah, or (becauſe A is as hh) as h^3, or (by ſimilar Δs) as VA^3.

COROLLARY X.

All ſimilar ſolids are as the cubes of their like ſides. This follows from their being compoſed of ſimilar pyramids.

EXAMPLE II.

Required the ſolidity of the ſquare pyramid each ſide of whoſe baſe is 30 and the ſlant height 25.



Firſt, $30\times30=900=$ the baſe.

But $BC=\sqrt{CA^2-AB^2}=\sqrt{25^2-15^2}$ $=\sqrt{5^2\times5^2-3^2\times5^2}=\sqrt{16\times5^2}=4\times5=$ $20=$ the perpendicular height.

Then $\frac{900\times20}{3}=300\times20=6000=$ the ſolidity.

EXAMPLE III.

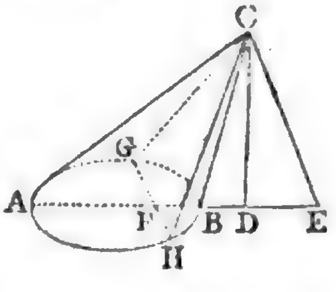
Being to meaſure an oblique cone; and having taken the circumference of the baſe equal to 40 feet, and the outward angle which the ſide of the cone, where it was ſhorteſt, made with the horizon 85°; going thence in a direct line towards the part to which the vertex inclined, to the diſtance of 100 feet, I found there the angle of elevation of the vertex of the cone to be 50°: required the ſolidity.

The $\angle B+\angle E=85°+50°=135°$.
And $180°-135°=45°=\angle BCE$.

Then

s. $\angle BCE=45°$ — 9·8494850
$BE=100$ — — 2·0000000
s. $\angle E=50°$ — 9·8842540
BC — — 2·0347690

And s. $\angle D=90°$ — — 10·0000000
BC — — — 2·0347690
s. $\angle B=85°$ — — 9·9983442
$DC=107·9228$ — 2·0331132



But $\frac{40}{3\cdot1416}=\frac{10}{\cdot7854}=12\cdot732396=$ the diameter.

And $12\cdot732396\times10=127\cdot32396=$ the baſe.

Whence $\frac{DC\times127\cdot32396}{3}=42\cdot44132\times107\cdot9228=$ $4580\cdot386=$ the content of the cone.

EXAMPLE IV.

There is a cone whoſe perpendicular altitude is 40 feet, and the circumference of its baſe 30 feet, which is cut by a plane drawn from the vertex, and paſſing through the baſe at the diſtance of 2 feet from the center; what is the ſolidity of the whole cone and of the two pyramids, *GHBC*, *GHAC*, into which it is cut? [The laſt fig.]

Now $\frac{30}{3\cdot1416}=9\cdot54929=$ the diameter of the baſe.

And $\frac{9\cdot54929}{2}=4\cdot774648=$ the radius.

Therefore $4\cdot774648-2=2\cdot774648=$ the verſed ſine of the leſs ſegment of the circle.

Then, to find its area by the table of circular ſegments, $2\cdot774648\div\frac{30}{3\cdot1416}=2\cdot774648\times\frac{3\cdot1416}{30}=2\cdot774648\times1\cdot0472=\cdot2905611$ the tabular verſed ſine, to which anſwers the tabular ſegment $\cdot18955709$, which multiplied by the ſquare of the diameter gives $17\cdot2856$ = the leſs ſegment, or baſe of the leſs pyramid.

But $\overline{\frac{30}{3\cdot1416}}\Big|^{2}\times\cdot7854=\frac{900}{4\times3\cdot1416}=71\cdot6197=$ the whole baſe.

And hence $71\cdot6197-17\cdot2856=54\cdot3341=$ the greater ſegment, or baſe of the greater pyramid.

Then

$$\text{Then } \tfrac{40}{3} \times \begin{cases} 17{\cdot}2856 = 230{\cdot}474\tfrac{2}{3} = \text{the leſs pyramid,} \\ 54{\cdot}3341 = \underline{724{\cdot}454\tfrac{2}{3}} = \text{the greater pyr.} \end{cases}$$

their ſum is $954{\cdot}929\tfrac{1}{3}$ = the whole cone.

Or, ſince 71·6197 is the area of the baſe of the cone, therefore $\frac{71{\cdot}6197 \times 40}{3} = 954{\cdot}929\tfrac{1}{3}$ = the ſolidity of the cone.

PROBLEM VIII.

To find the ſolidity of the fruſtum of a pyramid.

RULE.

Add into one ſum the areas of the two ends and the mean proportional between them, multiply the ſum by the perpendicular height, and $\frac{1}{3}$ of the product will be the ſolidity.

That is, If A be the area of the greater end,
a that of the leſs, and b the height,
Then $\overline{A+a+\sqrt{Aa}} \times \frac{1}{3} b$ will be the ſolidity.

Note 1. If the ends be regular polygons, the particular rule for them will be eaſier, thus: Add together the ſquare of a ſide of each end of the fruſtum, and the product of thoſe ſides, multiply the ſum into the height and the product into the tabular area anſwering to the particular figure of the ends, and $\frac{1}{3}$ of the laſt product will be the content.

Or, Divide the difference of the cubes of the ſaid ſides by their difference, and multiply the quotient by the height, and the tabular area, and take $\frac{1}{3}$ of the product.

Note 2. If the ends be circles, the fruſtum will be that of a cone, and then multiply ·2618 $\left(\frac{{\cdot}7854}{3}\right)$ by the height, and the product either by the quotient ariſing from the diviſion of the difference of the cubes of the diameters by the difference of the diameters, or by the ſum ariſing from the addition of the ſquare of each diameter and the product of the diameters.*

Ex-

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* All theſe rules come from the demonſtration of prob. 7 and its corollaries.

EXAMPLE I.

How many ſolid feet are there in a tree whoſe baſes are ſquares, each ſide of the one being 15 inches, and each ſide of the other 6, and the length meaſures along the ſide 24 feet? [Fig. to ex. 2 prob. 7.]

Here $5\times15=225=$ the greater baſe,
$6\times6=36=$ the leſs,
and $15\times6=\underline{90}=$ their mean.
their ſum is 351

But, $AB-DE=7\frac{1}{2}-3=4\frac{1}{2}=AF$, and $\sqrt{\overline{DA^2-AF^2}}$ $=\sqrt{\overline{24\times12|^2-4\frac{1}{2}|^2}}=287{\cdot}9649$ inches $=FD=$ the perpendicular height.

Therefore $\frac{351\times287{\cdot}9649}{3}=117\times287{\cdot}9649=33691{\cdot}8933$ inches $=19{\cdot}49762$ feet $=$ the ſolidity.

EXAMPLE II.

If a caſk which is two equal conic fruſtums joined together at the baſes, have its bung diameter 28 inches, its head diameter 20 inches, and length 40 inches; how many gallons of wine will it hold?

Here $20^2\times{\cdot}7854=314{\cdot}16=$ area at the end,
$28^2\times\underline{{\cdot}7854=615{\cdot}7536}=$ area of the bung circle,
and $\sqrt{\overline{20^2\times{\cdot}7854\times28^2\times7854}}=20\times28\times{\cdot}7854=439{\cdot}824$ $=$ their mean proportional; their ſum is $1369{\cdot}7376$, which multiplied by $\frac{40}{3}$ ($\frac{1}{3}$ of the length of both fruſtums together) produces $18263{\cdot}168$ ſolid inches; which divided by 231, the inches in a wine gallon,

$$\text{thus, } 231=3\times7\times11\left\{\begin{array}{r}3)\underline{18263{\cdot}168}\\ 7)\underline{6087{\cdot}7226}\\ 11)\underline{869{\cdot}6746}\end{array}\right.$$

gives $79{\cdot}0613$ wine gallons.

Or, $\overline{20^2+28^2+20\times28}\times40\times{}^{\cdot}2618=1744\times10{}^{\cdot}472$ $=18263{}^{\cdot}168=$ the ſolidity the ſame as before.

PROBLEM IX.

To find the ſolidity of a cuneus or wedge.

A wedge is a ſolid having a rectangular baſe *AC*, and two of the oppoſite ſides ending in an acies or edge *EF*.

RULE.

To twice the length of the baſe add the length of the edge, multiply the ſum by the product of the height of the wedge and breadth of the baſe, and $\frac{1}{6}$ of the laſt product will be the ſolidity.

That is, uſing the letters in the demonſtration below, $bh\times\frac{2L+l}{6}=$ the content.*

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* DEMONSTRATION.

Put $L=BC=$ the length of the baſe,
$l=EF=$ the length of the edge,
$b=BA=$ the breadth of the baſe,
$h=EP=$ the height of the wedge.

Then, ſince it is evident that, according as the edge is ſhorter or longer than the baſe, the wedge is greater or leſs than half a priſm of the ſame height and breadth with the wedge and length equal to that of the edge, by a pyramid of the ſame height and breadth at the baſe alſo, and the length of whoſe baſe is equal to the difference of the lengths of the edge and baſe of the wedge; we ſhall have $\frac{1}{2}blh\pm bh\times\frac{\pm L\mp l}{3}=\frac{1}{2}blh+bh\times\frac{L-l}{3}=bh\times\frac{3l+2L-2l}{6}=bh\times\frac{2L+l}{6}$. *Q.E.D.*

COROLLARY.

If $l=L$, the rule will become $bh\times\frac{3L}{6}=\frac{1}{2}bhL=\frac{1}{2}$ a priſm of the ſame baſe and height, as it ought.

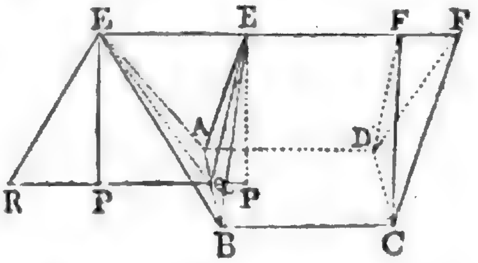
SCHOLIUM.

It is evident that whether the two ends or the two ſides of the wedge be equally or unequally inclined to the baſe, it will make no difference in the rule.

Note. If the length of the edge be equal to the length of the baſe, the wedge will be, evidently, equal to half a priſm of the ſame baſe and height, or equal to half the product of the baſe and height.

EXAMPLE I.

How many ſolid feet are in a wedge whoſe baſe is 3 feet 4 inches long and 10 inches broad, and each end is inclined to the baſe in an angle of 70°, the edge being 2 feet 6 inches long?



Firſt, $PQ = \frac{L-l}{2} = \frac{40-30}{2} = \frac{10}{2} = 5$ inches.

And, as radius : tan. $70° = L\ P :: PQ = 5 : 13{\cdot}737387 = PE = h$.

Then, $hb \times \frac{2L+l}{6} = \frac{10 \times 13{\cdot}737387 \times \overline{80+30}}{6} = 10 \times 110 \times 2{\cdot}2895645 = 1100 \times 2{\cdot}2895645 = 2518{\cdot}52095$ ſolid inches $= 1{\cdot}45747$ ſolid feet.

EXAMPLE II.

If the length and breadth of the baſe of a wedge be 35 and 15 inches, and the length of the edge 55; what will be its ſolidity in feet, ſuppoſing the baſe to make an angle of 115° with the one end of the wedge and of 125° with the other.

Let EP be perpendicular to the baſe, ER parallel to the end DCF, and RPQ perpendicular to AB.

Then $RQ = EF - BC = 55 - 35 = 20$, and the angles at R and Q = (the ſupplements of the angles 115° and 125°) 65° and 55°, and conſequently the angle $REQ = 60°$.

Hence, as s. $\angle E = 60^\circ$: s. $\angle Q = 55^\circ :: QR = 20 : RE = 18{\cdot}9175$; and as s. $\angle P = 90^\circ$: s. $\angle R = 65^\circ :: RE = 18{\cdot}9175 : EP = 17{\cdot}14508$ = the height of the wedge.

Wherefore $bh \times \frac{2L+l}{6} = 15 \times 17{\cdot}14508 \times \frac{70+55}{6} = 8{\cdot}57254 \times 5 \times 125 = 5357{\cdot}8375$ ſolid inches $= 3{\cdot}1006$ ſolid feet = the content required.

PROBLEM X.

To find the ſolidity of a priſmoid.

Definition.

A priſmoid is a ſolid having for its two ends any diſſimilar parallel plane figures of the ſame number of ſides, and all the ſides of the ſolid plane figures alſo.

It is evident that the ſides of this ſolid are all trapezoids.

If the ends of the priſmoid be bounded by curves, as ellipſes, &c. the number of its ſides or trapezoids will be infinite, and it is then called, ſometimes, a cylindroid.

GENERAL RULE.

To the ſum of the areas of the two ends add four times the area of a ſection parallel to and equally diſtant from both ends, multiply the laſt ſum by the height and ⅙ of the product will be the ſolidity.*

PAR-

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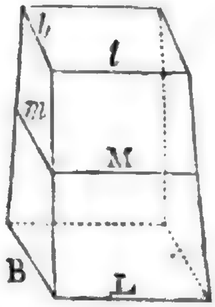
* It is evident that the rectangular priſmoid is compoſed of two wedges whoſe baſes are the two ends of the priſmoid, and whoſe heights are each equal to that of the priſmoid; wherefore, by the laſt problem, its ſolidity will be $= \overline{\overline{2L+l} \times B + \overline{2l+L} \times b} \times \frac{1}{6}h$. Which is the particular rule.

Co-

PARTICULAR RULE.

Or, If the baſes be diſſimular rectangles, take two correſponding dimenſions, and multiply each by the ſum of double the other dimenſion of the ſame end and the dimenſion of the other end correſponding to this other dimenſion; then multiply the ſum of the products by the height, and $\frac{1}{6}$ of the laſt product will be the ſolidity.



That is, if L, l be one dimenſion of each correſponding to each other, and B, b the other correſponding dimenſions, and h the height.

Then $\overline{\overline{2L+l}\times B+\overline{2l+L}\times b}\times\frac{1}{6}h=$ the ſolidity.

Note. Correſponding dimenſions are thoſe connected by a ſide of the ſolid, as is evident from the figure.

EXAMPLE I.

How many ſolid feet of timber are in a tree whoſe ends are rectangles, the length and breadth of the one

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COROLLARY I.

Since $\frac{L+l}{2}=M$, and $\frac{B+b}{2}=m$, are the length and breadth of a ſection parallel to and equally diſtant from each end, the above rule $\overline{\overline{2L+l}\times B+\overline{2l+L}\times b}\times\frac{1}{6}h$ or $\overline{2BL+Bl+2bl+bL}\times\frac{1}{6}h$, will become $\overline{BL+bl+4Mm}\times\frac{1}{6}h$. That is, the ſum of the areas of the two ends four times the ſection in the middle multiplied into $\frac{1}{6}h$.

COROLLARY II.

This laſt rule will ſerve for any priſmoid or cylindroid of whatever figures the ends may be, inaſmuch as they may be conceived to be compoſed of an infinite number of rectangular priſmoids. Which is the general rule.

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one being 14 and 12 inches, and their correſponding ſides of the other 6 and 4 inches; the perpendicular length being $30\frac{1}{2}$ feet?

First, by the general rule,

$$\left.\begin{array}{l}14\times12 = 168\\ 6\times4 = 24\end{array}\right\}\text{ areas of the ends}$$

$$\left.\begin{array}{l}\frac{14+6}{2}=\frac{20}{2}=10\\ \frac{12+4}{2}=\frac{16}{2}=8\end{array}\right\}\text{ the dimenſions in the middle}$$

$10\times8\times4=\underline{320}=$ four times the middle area, their ſum is 512 ſquare inches $=\frac{512}{144}$ ſquare feet $=\frac{32}{9}$ ſquare feet.

Then $\frac{32}{9}\times\frac{30\frac{1}{2}}{6}=\frac{16\times30\frac{1}{2}}{9\times3}=\frac{488}{3\times9}=\frac{162\frac{2}{3}}{9}=18\frac{2}{27}$ feet = the ſolidity.

Secondly, by the particular rule,

Here $L=14$, $B=12$, $l=6$, $b=4$ inches, and $h=30\frac{1}{2}$ feet; therefore $\overline{\overline{2L+l}\times B+\overline{2l+L}\times b}\times\frac{1}{6}h=\overline{\overline{28+6}\times12+\overline{12+14}\times4}\times\frac{30\frac{1}{2}}{6\times144}=\overline{17\times3+13}\times\frac{4\times30\frac{1}{2}}{3\times144}=64\times\frac{30\frac{1}{2}}{3\times36}=\frac{16\times30\frac{1}{2}}{3\times9}=\frac{488}{3\times9}=\frac{162\frac{2}{3}}{9}=18\frac{2}{27}$ feet, the ſame as before.

EXAMPLE II.

What is the capacity of a coal waggon whoſe inſide dimenſions are thus: at the top the length and breadth $81\frac{1}{2}$, and 55 inches; at the bottom, the length 41 and breadth $29\frac{1}{2}$; and the perpendicular depth $47\frac{1}{4}$.

Here $\overline{\overline{2L+l}\times B+\overline{2l+L}\times b}\times\frac{1}{6}h=$

$\overline{\overline{163+41}\times55+\overline{82+81{\cdot}5}\times29{\cdot}5}\times\frac{47{\cdot}25}{6}=$

$\overline{102\times55+81{\cdot}75\times29{\cdot}5}\times\frac{47{\cdot}25}{3}=8047{\cdot}75\times15{\cdot}75=$

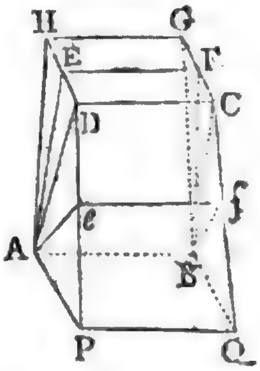
126752·0625 cubic inches = 449·4754 ale gallons = the content required, which is near one Newcaſtle chaldron of coals.

PROBLEM XI.

To find the ſolidity of the two Parts (commonly called ungulas or hoofs) into which the fruſtum of a rectangular or a ſquare pyramid, or a rectangular priſmoid, is cut by a plane inclined to its baſe.

CASE I.

If the plane, paſſing through *A* and *B*, cut the end in *EF*, between *GH* and *DC*; it will cut off the wedge *AGEFHB*, whoſe baſe is *GEFH*, edge *AB*, and height the ſame with that of the fruſtum, or priſmoid, and the remaining part *AEDCFBQP* will be a priſmoid.



Then, by problem 9, find the content of the wedge *ABHFEG*, and that of the priſmoid *ABQPDCFE* by problem 10.

EXAMPLE.

If the fruſtum of a ſquare pyramid be cut by a plane paſſing through one ſide of the leſs end and through the middle of the greater end; what are the contents of the two parts, ſuppoſing each ſide of the greater end to be 15 inches, each ſide of the leſs 6, and the ſlant height 24 feet?

Since by example 1 problem 8, the perpendicular height is 287·9649 inches, therefore,

Firſt, $\frac{2\times15+6}{6}\times7\frac{1}{2}\times287{\cdot}9649=45\times287{\cdot}9649=$ $12958{\cdot}4205$ inches $=7{\cdot}499086$ feet $=$ the content of the wedge.

And $\overline{\overline{30+6}\times 7\tfrac{1}{2}+\overline{12+15}\times 6}\times\frac{287\cdot9649}{6} = \overline{18\times15+27\times6}\times\frac{287\cdot9649}{6} = 72\times287\cdot9649 = 20733\cdot4728$ inches = 11·998536 feet = the priſmoid.

CASE II.

If the plane paſs through *D* and *C* as well as *A* and *B*, the ſolid is thereby divided into two wedges or hoofs *AGDCHB*, *DPABQC*, whoſe two baſes are the ends or baſes of the ſolid.

And then the contents of the two parts will be found by problem 9.

EXAMPLE.

Let there be taken here the ſame figure as in the laſt example, to find the content of the two wedges into which it is cut.

Firſt, $\frac{30+6}{6}\times15\times287\cdot9649=90\times287\cdot9649=25916\cdot841$ inches = 14·99817 feet = the greater wedge.

And $\frac{12+15}{6}\times6\times287\cdot9649 = 27\times287\cdot9649 = 7775\cdot0523$ inches = 4·49945 feet = the leſs wedge.

CASE III.

If the plane cut the ſide of the ſolid in *ef*; the part cut off *ePABQf* will be a wedge whoſe baſe is *AQ*, and which being taken from the whole figure will leave the content of the part *ABfeDCHG*.

EXAMPLE.

Let there be taken here the ſame figure, ſuppoſing the plane to cut the ſide *PC* in *ef* at the diſtance of 10

10 feet from PQ; to find the ſolidity of the two parts.

Firſt, as $PD = 24 : Pe = 10 :: 287{\cdot}9649$ inches = the height of the whole ſolid : $119{\cdot}98536$ inches = the height of the wedge AQe. And as $PD = 24 : PE = 10 :: DC - PQ = 9 : ef - PQ = \frac{15}{4} = 3{\cdot}75$; hence $ef = 3{\cdot}75 + 6 = 9{\cdot}75$ = the length of the edge of the cuneus. Wherefore $\frac{12 + 9{\cdot}75}{6} \times 6 \times 119{\cdot}98536 = 21\frac{3}{4} \times 119{\cdot}98536 = 87 \times 29{\cdot}99634 = 2609{\cdot}68158$ inches $= 1{\cdot}51023$ feet = the ſolidity of the wedge AQe.

Which taken from $19{\cdot}49762$ = the content of the whole fruſtum found by problem 9, leaves $17{\cdot}98739$ for the content of the part $ABfeDGHC$.

PROBLEM XII.

To find the ſolidity of the elliptic hoofs of the fruſtum of a cone, made by a plane cutting diagonally the oppoſite extremities of the ends.

* RULE.

From the ſquare of the diameter of the baſe of the hoof, ſubtract the product of the diameter of the other

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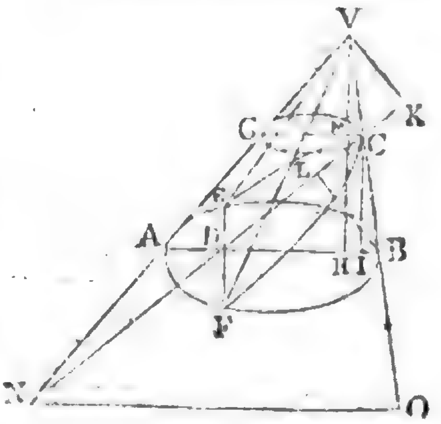
* Let $AEBF$ be the baſe of a cone, or of any other pyramid, right, or oblique; AVB a ſection through the vertex by a plane perpendicular to the baſe; and VFE, ECF, two other ſections perpendicular to AVB, the former through the vertex and the latter through the ſide, at C, between V and B. On AB let fall the perpendiculars VH, CI; and on DC the perpendiculars VK, BL; and draw CG parallel to AB, and meeting AV and VH in G and M.



Then

other end of the fruſtum and a mean proportional between the diameters; divide the difference by the difference of the diameters; multiply the quotient by the height of the hoof and the product by the diameter of its baſe; ſo ſhall the laſt product multiplied by $\cdot 2618$ $\left(\frac{\cdot 7854}{3}\right)$ produce the content of the hoof.

That is, Putting $D=AB$ the diameter of the greater end,

$d=CG$ the leſs diameter,

and $b=CI$ the perpendicular height, or diſtance of the baſe and end.

Then $\frac{D^2-d\sqrt{Dd}}{D-d}\times 0{\cdot}2618\,db=$ the greater hoof ABC.

And $\frac{D\sqrt{Dd}-d^2}{D-d}\times 0{\cdot}2618\,db=$ the leſs or complemental hoof ACG. A plane being conceived to be drawn through C and A.—The proof in corollary 2.

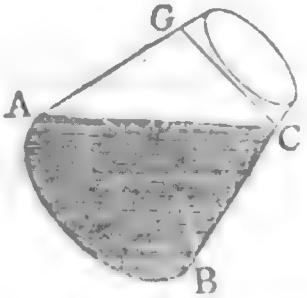
Then it will be evident that $EFBV$ is a pyramid whoſe baſe is EFB (A), altitude VH (a); and therefore its content is equal to $\frac{1}{3}Aa$; and that $EFCV$ is a pyramid whoſe baſe is EFC (B), height VK (b), and therefore its content equal to $\frac{1}{3}Bb$; and moreover that the difference of theſe pyramids, or $\frac{1}{3}Aa-\frac{1}{3}Bb$ is the content of the hoof $EFBC$.

But, by the ſimilar triangles ABV, VCG, it will be $AB-GC:CI$ $(HV-VM)::AB:HV\,(a)=\frac{AB\times CI}{AB-CG}$; alſo $AB-GC:CI$ $(::AB:HV)::GC:VM=\frac{GC\times CI}{AB-CG}$; and $CD:DB::$ (by the ſimilar triangles ICD, DBL) $CI:BL::$ (becauſe of the ſimilar triangles BCI and CMV, VKC and CLB) $MV\left(=\frac{GC\times CI}{AB-CG}\right):VK\,(b)=\frac{GC\times CI\times BD}{DC\times\overline{AB-CG}}$.

Wherefore the hoof $EFBC$ will be $=\frac{\frac{1}{3}CI}{AB-CG}\times\overline{A\times AB-B\times\frac{GC\times BD}{DC}}$, which is a general theorem for any hoof of any pyramid.

EXAMPLE I.

If a conical veſſel whoſe bottom diameter is 30 inches, be inclined to the horizon till the liquor in it juſt cover the bottom, and its ſurface, diſpoſing itſelf into the poſition *AC*, cut the ſide *BC* of the veſſel in *C* at the diſtance of 18 inches from the bottom *AB*; how many wine gallons of liquor are in it, ſuppoſing the diameter *GC* of the veſſel at *C* to be 19·2 inches?



Here $D=30$, $d=19{\cdot}2$, and $h=18$.

Whence $\frac{D^2-d\sqrt{Dd}}{D-d}\times{\cdot}2618\,Dh=$
$\frac{30^2-19{\cdot}2\sqrt{30\times19{\cdot}2}}{30-19{\cdot}2}\times30\times18\times{\cdot}2618=$
$\frac{439\frac{1}{5}\times30\times18\times{\cdot}2618}{10\frac{4}{5}}=\frac{2196\times30\times18\times{\cdot}2618}{54}$
$=2196\times{\cdot}2618\times10=5749{\cdot}128$ inches. Which being divided by 231

thus, $231=3\times7\times11\left\{\begin{array}{l} 3)\underline{5749{\cdot}128} \\ 7)\underline{1916{\cdot}376} \\ 11)\underline{273{\cdot}768} \end{array}\right.$

gives 24·888 wine gallons for the content required.

Ex-

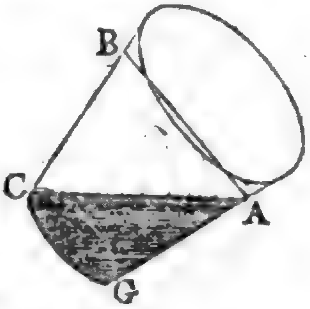
COROLLARY I.

If the baſe be circular, or the pyramid a cone, and the angle *CDB* be leſs than the angle *VAD*; or, which is the ſame, if *CD* and *VA*, produced, interſect in *N*; the ſection *ECF* will be a ſegment of an ellipſe whoſe tranſverſe axe is *CN*, and conjugate $\sqrt{NO\times CG}$, *NO* being drawn parallel to *AB* and meeting *VB*, produced, in *O*. And then the above general theorem will become $\frac{\frac{1}{3}CI}{AB-CG}\times:AB\times$ circular ſeg-

ment

EXAMPLE II.

Let there be taken here the ſame dimenſions as in the laſt example, ſuppoſing the veſſel to be narroweſt at the bottom, to find how many ale gallons are in it.



Here $\frac{D\sqrt{Dd-dd}}{D-d}\times{}^{\cdot}2618\,dh=$ $\frac{30\sqrt{30\times19\frac{1}{5}-19{}^{\cdot}2^2}}{30-19\frac{1}{5}}\times{}^{\cdot}2618\times19\frac{1}{5}\times18=$ $\frac{351\frac{9}{25}\times19\frac{1}{5}\times18\times{}^{\cdot}2618}{10\frac{4}{5}}=\frac{351\frac{9}{25}\times96\times18\times{}^{\cdot}2618}{54}$ $=117\frac{3}{25}\times96\times{}^{\cdot}2618=117{}^{\cdot}12\times96\times$ ${}^{\cdot}2618=2943{}^{\cdot}55$ inches. Which being divided by 282, give 10ˈ43 for the number of ale gallons required

PRO-

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ment $EBF-\frac{CG\times BD}{DC}\times$ elliptic ſegment $ECF=$ (by prob. 6 ſect. 3 part 3) $\frac{\frac{1}{3}CI}{AB-CG}\times:AB\times$ cir. ſeg. $EBF-\frac{CG\times BD\times\sqrt{NO\times CG}}{DC\times CN}\times$ cir. ſeg. whoſe diameter is CN and height $CD=$ (ſince ſimilar ſegments are as the ſquares of their diameters) $\frac{\frac{1}{3}CI}{AB-CG}\times:AB\times$ circular ſegment $EBF-$ $\frac{CG\times BD\times CN\times\sqrt{NO\times CG}}{DC\times AB^2}\times$ ſeg. of the cir. $AEBF$ whoſe height is $\frac{AB\times DC}{CN}$ $=$ the elliptic hoof $EFCB$.

But, by ſim. Δs, $CG-AD:DC::CG:CN=\frac{GC\times CD}{CG-AD}$, and $CG-AD:DB::CG:NO=\frac{GC\times DB}{CG-AD}$; which values of NO and CN being ſubſtituted inſtead of them in the above expreſſion of the elliptic hoof, will give $\frac{\frac{1}{3}CI}{AB-CG}\times:AB\times$ cir. ſeg. $EBF-\frac{CG^3}{AB^2}\times\overline{\frac{BD}{CG-AD}}\Big|^{\frac{1}{2}}\times$ ſeg. of the cir. $AEBF$ whoſe height is $\frac{AB\times\overline{CG-AD}}{CG}=\frac{\frac{1}{3}h}{D-d}\times D\times$ circular ſeg. EBF

PROBLEM XIII.

To find the ſolidity of the elliptic hoofs of the fruſtum of a cone made by a plane cutting off a part of the baſe.

RULE.

1. From the verſed ſine or height of the baſe of the hoof ſubtract the difference of the diameters of the baſe and end of the fruſtum, and divide the remainder by the diameter of the end of the fruſtum; find the tabular ſegment whoſe verſed ſine is equal to the quotient; find alſo the tabular ſegment whoſe verſed ſine is expreſſed by the quotient of the verſed ſine of the baſe of the hoof divided by the diameter of the baſe of the fruſtum; multiply the former ſegment by the cube of the diameter of the end, by the quotient of the verſed ſine of the baſe of the hoof divided

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$EBF - \frac{d^3}{D^3} \times \overline{\frac{BD}{BD - \overline{D-d}}}\Big|^{\frac{3}{2}} \times$ ſegment of the circle AB whoſe height is $\frac{D \times \overline{BD - \overline{D-d}}}{d} =$ the elliptic hoof $EFCB$; putting h for the height of the hoof, and D and d for the diameters of the baſe and end or top of the fruſtum reſpectively.

Or, if we would reduce the circular ſegments in the laſt expreſſion to thoſe of a circle whoſe diameter is 1, to be found in the table of circular ſegments at the end of the book, that expreſſion will become $\frac{\frac{1}{3}h}{D-d} \times : D^3 \times$ tab. ſeg. whoſe height is $\frac{BD}{D} - d^3 \times \overline{\frac{BD}{BD - \overline{D-d}}}\Big|^{\frac{3}{2}} \times$ tab. ſeg. whoſe height is $\frac{BD - \overline{D-d}}{d} =$ the elliptic hoof $EFCB$. And this is the rule in prob. 13.

And if this value of the hoof be taken from $\left(\frac{1}{3} hn \times \frac{D^3 - d^3}{D-d} =\right)$ that of the whole conic fruſtum, the remainder $\frac{\frac{1}{3}h}{D-d} \times : \overline{D^3 - d^3} \times n -$ D^3

divided by the difference between the ſaid verſed ſine and the difference of the diameters, and by the root of the ſaid quotient; and multiply the latter ſegment by the cube of the greater diameter; multiply the difference of theſe two products by $\frac{1}{3}$ of the height of the hoof, and the product divided by the difference of the diameters will give the content of the hoof required.

2. Subtract the meaſure of the hoof, found above, from that of the whole fruſtum, and the remainder will be the meaſure of the complemental hoof.

$D^3 \times$ tab. ſeg. whoſe height is $\frac{BD}{D} + d^3 \times \left.\overline{\frac{BD}{BD-\overline{D-d}}}\right|^{\frac{1}{2}} \times$ tabular ſeg. whoſe height is $\frac{BD-\overline{D-d}}{d}$ or $\frac{\frac{1}{3}h}{D-d} \times : -nd^3 + D^3 \times$ tab. ſeg. whoſe height is $\frac{AD}{D} + d^3 \times \left.\overline{\frac{BD}{BD-\overline{D-d}}}\right|^{\frac{1}{2}} \times$ tab. ſeg. whoſe height is $\frac{BD-\overline{D-d}}{d}$ will expreſs the meaſure of the complemental elliptic hoof *EFCGA*.

COROLLARY II.

If the points D and A coincide, the ſection EFC will be a whole ellipſe, and the rules in corollary 1 will become $\frac{1}{3}Dhn \times \frac{DD-d\sqrt{Dd}}{D-d}$ $=$ the elliptic hoof ACB, and $\frac{1}{3}dhn \times \frac{D\sqrt{Dd}-dd}{D-d} =$ the complemental elliptic hoof ACG. And theſe are the rules in prob. 12.

COROLLARY III.

Since $\frac{h}{D-d} = \frac{H}{D}$ (H denoting the height of the whole cone), the laſt theorems will become $\frac{1}{3}nH \times \overline{D^2-d\sqrt{Dd}} = ACB$, and $\frac{1}{3}nH \times \overline{D\sqrt{Dd}-dd}$ $= ACG$.

COROLLARY IV.

But $\frac{1}{3}nHD^2 =$ the whole cone VAB, and therefore $\frac{1}{3}nHD^2 - \frac{1}{3}nH \times \overline{D^2-d\sqrt{Dd}} = \frac{1}{3}nHd\sqrt{Dd} =$ the top part VAC. Which top part, conſe-

That is, if D and d be the extreme diameters of the fruſtum, and b its height; alſo P = the tabular ſegment whoſe verſed ſine is $\frac{BD}{D}$, $\mathcal{Q}$ = that whoſe verſed ſine is $\frac{BD-\overline{D-d}}{d}$, and $n = \cdot785398$ &c.

Then $\overline{P\times D^3 - \mathcal{Q}\times d^3\times\frac{BD}{BD-\overline{D-d}}\sqrt{\frac{BD}{BD-\overline{D-d}}}}\times\frac{\frac{1}{3}b}{D-d} =$ the content of the elliptic hoof $EFCB$.

And $\overline{\overline{D^3-d^3}\times n - P\times D^3 + \mathcal{Q}\times d^3\times\frac{BD}{BD-\overline{D-d}}\sqrt{\frac{BD}{BD-\overline{D-d}}}}\times\frac{\frac{1}{3}b}{D-d}$ = to that of the complemental hoof $EFCGA$. —Theſe are proved in corollary 2 to prob. 12.—See the fig. in page 168.

Ex-

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conſequently, is to the whole cone VAB as ($\frac{1}{3}nHd\sqrt{Dd}$ to $\frac{1}{3}nHD^2$ as) $d^{\frac{3}{2}}$ to $D^{\frac{3}{2}}$; or the ſquare of the whole cone VAB is to the ſquare of the top part VAC, as D^3 to d^3.

COROLLARY V.

If the angle CDB be equal to the angle VAB, or if the ſection ECF be parallel to the ſide AG of the cone, it will be a parabola whoſe axe is CD, baſe $EF = 2\sqrt{\overline{BD}\times\overline{DA}} = 2\sqrt{\overline{D-d}\times d}$, and its area (by prob. 5 ſect. 4 part 3) $= \frac{2}{3}CD\times EF = \frac{4}{3}CD\sqrt{\overline{D-d}\times d}$; and therefore the gen. theor. will become $\overline{\frac{\frac{1}{3}b}{D-d}\times D\times\text{cir. ſeg. } EBF - \frac{d\times\overline{D-d}}{CD}\times\frac{4}{3}CD\sqrt{\overline{D-d}\times d}}$ $= \frac{1}{3}b\times\overline{\frac{D^3}{D-d}\times\text{tab. ſeg. whoſe height is }\frac{D-d}{d} - \frac{4}{3}d\sqrt{\overline{D-d}\times d}}$ = the parabolic hoof $EFBC$.

Which being taken from $\frac{1}{3}bn\times\frac{D^3-d^3}{D-d}$ = the meaſure of the whole fruſtum $ABCG$, the remainder $\frac{1}{3}b\times : \frac{4}{3}d\sqrt{\overline{D-d}\times d} - \frac{nd^3}{D-d} + \frac{D^3}{D-d}\times$ tab. ſeg. whoſe height is $\frac{d}{D}$ or $\frac{\frac{1}{3}b}{D-d}\times : \overline{D-d}\times\frac{4}{3}d\sqrt{d\times\overline{D-d}} - nd^3 + D^3\times$ tab. ſeg. whoſe height is $\frac{d}{D}$ will expreſs the meaſure of the complemental parabolic hoof $EFCGA$.

Co-

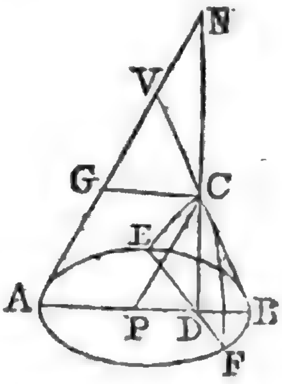
EXAMPLE I.

If a veſſel in the form of the fruſtum of a cone, whoſe bottom diameter is 30 inches, be inclined to the horizon till the ſurface of the liquor in it cut the bottom, leaving 10 inches of its diameter dry, and meeting with the ſide in *C* at the diſtance *CI* of 18 inches from the baſe: How many Wincheſter gallons are in it, ſuppoſing the diameter *CG* of the veſſel at the top of the liquor to be $19\frac{1}{5}$ inches?

Here $D=30$, $d=19\frac{1}{5}$, $b=18$, and $BD=30-10=20$. Whence $\frac{BD}{D}=\frac{20}{30}=\frac{2}{3}=\cdot6666\frac{2}{3}$, the tabular area correſponding to which is $\cdot55622573$; and $\frac{BD-\overline{D-d}}{d}=\frac{20-\overline{30-19\frac{1}{5}}}{19\frac{1}{5}}=\frac{20-10\frac{4}{5}}{19\frac{1}{5}}=\frac{9\frac{1}{5}}{19\frac{1}{5}}=\frac{46}{96}=\frac{23}{48}=\frac{2\cdot875}{6}=\cdot479\frac{1}{6}$, the tabular area anſwering to which is $\cdot37187178$.

Again,

COROLLARY VI.



If the $\angle CDB$ exceed the $\angle VAB$, or, which is the ſame thing, if *CD* and *VA*, produced, interſect above *V*, in *N*, the ſection *ECF* will be an hyperbola, whoſe tranſverſe axe is *CN*. But in the hyperbola, as well as in the other conic ſections, the tranſverſe axe $CN=\frac{GC\times CD}{DP}$ *CP* being drawn parallel to *VA*) $=\frac{d\times CD}{D-d-BD}$, and the conjugate of each alſo $=GC\sqrt{\frac{BD}{DP}}=d\sqrt{\frac{BD}{D-d-BD}}$; and the area of the hyperbolic ſection being found by prob. 6 ſect. 5 part 3, and ſubſtituted in the general theorem, will give the ſolidity of the hyperbolic ungula.

Co-

Again, $D^3=27000$, which multiplied into ·55622573, the former area, produces 15018·09471, and $d^3 \times \frac{BD}{BD-\overline{D-d}} \times \sqrt{\frac{BD}{BD-\overline{D-d}}} = \overline{19\frac{1}{5}}|^3 \times \frac{20}{9\frac{1}{5}} \times \sqrt{\frac{20}{9\frac{1}{5}}} = 19\frac{1}{5} \times \overline{\frac{19\frac{1}{5}}{9\frac{1}{5}}}|^2 \times 20\sqrt{184} = \frac{19\frac{1}{5} \times 48^2 \times 40}{23 \times 23}\sqrt{46} = 22686·470698$, which multiplied by ·37187178, the latter area, gives 8436·45824.

Then 6581·63647, the difference of theſe two products, being multiplied by $\frac{6}{10\frac{4}{5}} = \frac{30}{54} = \frac{5}{9} = \frac{\frac{1}{3}h}{D-d}$ = the quotient of ⅓ of the height divided by the difference of the diameters, will produce 3656·4647 for the ſolidity in inches; which being divided by 268⅘, the inches in a corn gallon, give 13·6029 corn or wincheſter gallons for the quantity of liquor in the veſſel.

EXAMPLE II.

If a veſſel, in form of the fruſtum of a cone, cloſe at both ends, be placed in ſuch poſition that the liquor juſt covers the leſs end and 10 inches of the diameter of the greater; what number of wine gallons are in it, ſuppoſing the diameters of the two ends to be 30 and 19⅕ inches, and their diſtance 18 inches?

Here the part filled is the complemental hoof to that in the laſt example. Wherefore if from D^3+

COROLLARY VII.

If D coincide with I, or CDB be a right angle, the tranſverſe and conjugate axes of the hyperbolic ſection will become $\frac{2dh}{D-d}$ and d reſpectively.

$\overline{D^2+d\times\overline{D+d}}\times\frac{1}{3}nh=\overline{30^2+19\frac{1}{3}\times49\frac{1}{3}}\times6\times785398$ &c. = 8692·661, the content of the whole fruſtum, be taken 3656·4647, that of the hoof in the laſt example, the remainder 5036·1963 will be the content, in inches, of the complemental hoof; which being divided by 231, will give 21·8018 wine gallons for the content required.

PROBLEM XIV.

To find the parabolic hoofs of the fruſtum of a cone; that is, of the hoofs made by a plane paſſing parallel to the ſide of the cone.

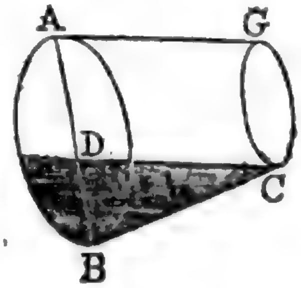
RULE.

Multiply the baſe of the hoof by the greater diameter of the fruſtum, and divide the product by the difference of the diameters; from the quotient ſubtract $\frac{2}{3}$ of the product ariſing from the multiplication of the leſs diameter by the ſquare root of the product of the leſs diameter and difference of the diameters; then the remainder multiplied by $\frac{1}{3}$ of the height will be the content of the hoof required.

That is, $\overline{\frac{A\times D}{D-d}-\frac{2}{3}d\sqrt{\overline{D-d}\times d}}\times\frac{1}{3}h=$ the parabolic hoof *EFCB*; where *D* and *d* are the diameters *AB* and *CG*, of the fruſtum, *h* the height *CI*, and *A* the baſe *EBF* of the hoof.—See the figure in page 168.

EXAMPLE.

If a veſſel in the form of the fruſtum of a cone be laid with its upper ſide parallel to the horizon, and if the greateſt diſtance of the ſurface of the liquor from the bottom, whoſe diameter is 30 inches, be 18 inches; how many ale gallons of liquor are in it, ſuppoſing the diameter of the veſſel at the greateſt diſtance of the ſurface of the liquor from the bottom to be $19\frac{1}{5}$ inches.



Here the dimenſions are the ſame as in the laſt problems; and $BD = D - d = 30 - 19\frac{1}{5} = 10\frac{4}{5} = 10{\cdot}8$ the verſed ſine or height of the baſe; which divided by the diameter 30, gives ·36 = the tabular verſed ſine, the area anſwering to which is ·25455056; which multiplied by 900, the ſquare of the diameter, gives 229·095504 for the circular ſegment or baſe of the hoof. Then $229{\cdot}0955 \times \frac{30}{10{\cdot}8} = \frac{2290{\cdot}955}{3{\cdot}6} = \frac{3818{\cdot}25834}{6} = 636{\cdot}37639$.

Again $\frac{4}{3}d\sqrt{\overline{D-d} \times d} = \frac{4}{3} \times 19\frac{1}{5}\sqrt{10\frac{4}{5} \times 19\frac{1}{5}} = \frac{4}{3} \times 19{\cdot}2 \times \sqrt{\frac{54}{5} \times \frac{96}{5}} = \frac{4}{3} \times 19{\cdot}2 \times 14{\cdot}4 = 4 \times 6{\cdot}4 \times 14{\cdot}4 = 368{\cdot}64$.

Then $\overline{636{\cdot}37639 - 368{\cdot}64} \times \frac{1}{3} = 267{\cdot}73639 \times 6 =$ 1606c·41934 inches, equal to the ſolidity; which being divided by 282, give 5·696526 ale gallons for the quantity of liquor required.

Note 1. If from 8692·661, the content of the whole fruſtum, found in the ſecond example of the laſt problem, be taken 1606·419, the content of the hoof, above found, the remainder, 7086·242, will be that of the complemental parabolic hoof $DAGC$.

Note 2. If the ſection EGF be an hyperbola, the hoofs may then be found after the manner deſcribed in the ſixth corollary to prob. 13.

PRO-

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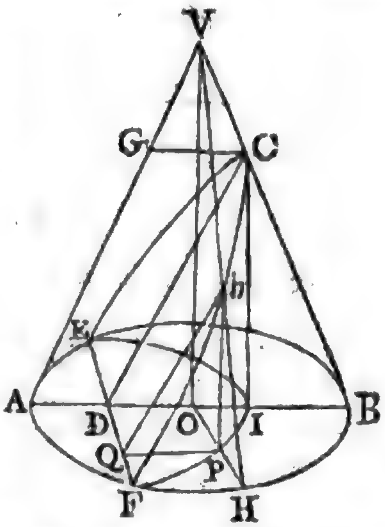
PROBLEM XV.

* *To find the convex ſurface of the elliptic hoofs of the fruſtum of a cone, made by a plane cutting the oppoſite extremities of the ends.*

RULE.

To four times the ſquare of the height of the fruſtum add the ſquare of the difference of the diameters of the baſe and end, divide the ſquare root of the ſum by the difference of the ſaid diameters, and call the quotient Q.

Multiply



* Let the right-angled triangle VOH be conceived to revolve about the axe VO, the part Hb of the hypothenuſe deſcribing the ſurface of the ungula, whilſt the part HP of the radius of the baſe of the cone, cut off by the perpendicular bP, deſcribes the ſpace $FPIEBF$. —Then ſince, by the ſimilar △s VOH, HbP, it is every-where as $PH : Hb :: OB\ (OH) : BV\ (HV)$, the ſpaces generated by theſe lines will be, likewiſe, in the ſame ratio, viz. $OB : BV ::$ the area $FIEBF$: the ſurface $FCEB\ (S) = \frac{VB}{BO} \times FIEBF$.—And in the ſame manner, by reaſon of the ſimilar △s DIC, QPb, (QP being drawn parallel to BD) we ſhall have $CD : DI ::$ $FCEF$ ($= B =$ the ſpace deſcribed by bQ) : $FIEF$ (the ſpace deſcribed in the ſame time by QP) $= \frac{ID}{DC} \times FCEF = \frac{ID}{DC} \times B$.—But the ſpace $FIEBF =$ the circular ſeg. $FBEF\ (A) - FIEF = A - \frac{ID}{DC} \times B$. And conſequently $S = \frac{VB}{BO} \times FIEBF = \frac{VB}{BO} \times \overline{A - \frac{ID}{DC} \times B} = \frac{CB}{BI} \times \overline{A - \frac{ID}{DC} \times B} =$ the convex ſurface of the ungula $EFBC$.

And

Multiply half the ſum of the ſaid diameters by the ſquare root of their product, and call the product P. Then

1. Q multiplied by P and by ·785398 &c. will produce the convex ſurface of the oblique cone ACV.

That is, $\frac{n\sqrt{4b^2+\overline{D-d}|^2}}{D-d} \times \frac{D-d}{2}\sqrt{Dd} =$ the curve ſurface of ACV.

2. Q multiplied by the difference between P and the ſquare of the diameter of the baſe, and by ·785398 &c. will produce that of the hoof, ABC, including the baſe.

That is, $\frac{n\sqrt{4b^2+\overline{D-d}|^2}}{D-d} \times \overline{DD-\frac{D+d}{2}\sqrt{Dd}} =$ the curve ſurface of ABC.

3. Q

And the method of proceeding will be the ſame for any other kind of pyramid.

COROLLARY I.

If from $\frac{VB}{BO} \times$ the circle AB, the convex ſurface of the whole cone, be taken that of the ungula, above found, the remainder $\frac{VB}{BO} \times \overline{FAEF + \frac{ID}{DC}} \times FCEF$, will expreſs the convex ſurface of the remaining part, $EFCVA$, of the cone.

COROLLARY II.

And if from the value of the part laſt found be taken $\frac{VB}{BO} \times$ circle CG $(= \frac{VC}{OI} \times$ circle $CG)$ the convex ſurface of the top cone CVG, the remainder, $\frac{VB}{BO} \times \overline{FAEF + \frac{ID}{DC} \times FCEF -}$ circle CG, will expreſs that of the complemental ungula $EFAGC$.

Co-

3. *Q* multiplied by the difference between *P* and the ſquare of the diameter of the top, and by ·785398 &c. will produce that of the hoof, *ACG*, including the top.

That is, $\frac{n\sqrt{4b^2+\overline{D-d}|^2}}{D-d}\times\overline{\frac{D+d}{2}\sqrt{Dd-dd}}$ = the curve ſurface of *ACG*.

Where the letters denote the ſame Quantities as in the foregoing problems.—All theſe are proved in corollary 4.

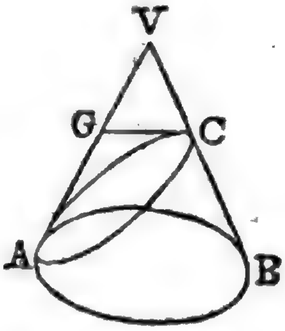
COROLLARY III.

If the $\llcorner CDB$ be leſs than the $\llcorner GAB$, or the ſect. *FCE* be an ellipſe, the ſurface of the ungula will be $\frac{VB}{BO}\times$ cir. ſeg. $FBE-\frac{ID}{DC}\times$ ell. ſeg. FCE = (by proceeding as in cor. 1 prob. 12) $\frac{VB}{BO}\times:EBF-\frac{CG^2\times ID}{AB^2\times\overline{CG-AD}}\sqrt{\frac{DB}{CG-AD}}\times$ ſeg. of the circle *AB* whoſe height is $AB\times\frac{CG-AD}{CG}=\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d}\times:EBF-\frac{dd}{DD}\times\frac{BD-\frac{1}{2}\times\overline{D-d}}{BD-\overline{D-d}}\sqrt{\frac{BD}{BD-\overline{D-d}}}\times$ ſeg. of the circle *AB* whoſe height is $D\times\frac{BD-\overline{D-d}}{d}=\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d}\times:D^2\times$ tab. ſeg. whoſe height is $\frac{BD}{D}-d^2\times\frac{BD-\frac{1}{2}\times\overline{D-d}}{BD-\overline{D-d}}\sqrt{\frac{BD}{BD-\overline{D-d}}}\times$ tab. ſeg. whoſe height is $\frac{BD-\overline{D-d}}{d}$. Where *b* is the height, and *D* and *d* the diameters of the baſe and top of the fruſtum.

And the value of the ſurface of the remaining part *EFCVA*, in corollary 1, will become $\frac{VB}{BO}\times:$ cir. ſeg. $FAE+\frac{ID}{DC}\times$ ellip. ſeg. $FCE=\frac{VB}{BO}\times:FAE+\frac{CG^2\times ID}{AB^2\times\overline{CG-AD}}\sqrt{\frac{DB}{CG-AD}}\times$ ſeg. of the circle *AB* whoſe

EXAMPLE.

Required the convex ſurfaces of the two hoofs of a conic fruſtum whoſe baſe and top diameters are 30 and $19\frac{1}{5}$, and height 18 inches, the ſection being made through the contrary extremities of the diameters; together with that of the oblique cone ACV.



Here $D=30$, $d=19\frac{1}{5}$, and $b=18$.

Whence $Q=\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d}=$ $\frac{\sqrt{4\times18^2+10{\cdot}8^2}}{10{\cdot}8}=\frac{\sqrt{4^2+1{\cdot}2^2}}{1{\cdot}2}=\frac{1}{3}\sqrt{10^2+3^2}$ $=\frac{1}{3}\sqrt{109}=3{\cdot}48010217$.

$P=\frac{D+d}{2}\sqrt{Dd}=24\frac{3}{5}\sqrt{30\times19\frac{1}{5}}=$ $24\frac{3}{5}\times24=590\frac{2}{5}$.

And

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whoſe height is $AB\times\frac{CG-AD}{CG}=\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d}\times:FAE+\frac{dd}{DD}\times$ $\frac{BD-\frac{1}{2}\times\overline{D-d}}{BD-\overline{D-d}}\sqrt{\frac{BD}{BD-\overline{D-d}}}\times$ ſeg. of the cir. AB whoſe height is $D\times$ $\frac{BD-\overline{D-d}}{d}=\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-a}\times:D^2\times$ tab. ſeg. whoſe height is $\frac{AD}{D}+$ $d^2\times\frac{BD-\frac{1}{2}\times\overline{D-d}}{BD-\overline{D-d}}\sqrt{\frac{BD}{BD-\overline{D-d}}}\times$ tab. ſeg. whoſe height is $\frac{BD-\overline{D-d}}{d}$.

Or it is $=\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d}\times:FAE+\frac{dd}{DD}\times\frac{\frac{1}{2}\times\overline{D+d}-AD}{d-AD}\sqrt{\frac{D-AD}{d-AD}}$ $\times$ ſeg. of the cir. AB whoſe height is $D\times\frac{d-AD}{d}=\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d}$ $\times:D^2\times$ tab. ſeg. whoſe height is $\frac{AD}{D}+d^2\times\frac{\frac{1}{2}\times\overline{D+d}-AD}{d-AD}\sqrt{\frac{D-AD}{d-AD}}$ $\times$ tab. ſeg. whoſe height is $\frac{d-AD}{d}$

Alſo

And then

1. $Q \times P \times \cdot 785398 = 2\cdot 73326528 \times 590\cdot 4 = 1613\cdot 7198213$ inches $= 11\cdot 2063876$ feet $=$ the convex ſurface of the oblique cone ACV.

2. $Q \times \cdot 785398 \times \overline{D^2 - P} = \cdot 785398 \times 3\cdot 48010217 \times 900 - 590\cdot 4 = 2\cdot 73326528 \times 309\cdot 6 = 846\cdot 2189307$ inches $= 5\cdot 87652$ feet $=$ that of the hoof ABC.

3. $Q \times \cdot 785398 \times \overline{P - d^2} = \cdot 785398 \times 3\cdot 48010217 \times 509\cdot 4 - 368\cdot 64 = 2\cdot 73326528 \times 221\cdot 76 = 606\cdot 1289085$ inches $= 4\cdot 2092285$ feet $=$ that of the complemental hoof ACG.

PRO-

Alſo that of the complemental hoof in cor. 2, will become $\frac{VB}{BO} \times :$ cir. ſeg. $FAE + \frac{ID}{DC} \times$ ellip. ſeg. $FCE -$ the cir. $CG = \frac{VB}{BO} \times : - n \times CG^2 + FAE + \frac{CG^2 \times ID}{AB^2 \times \overline{CG - AD}} \sqrt{\frac{DB}{CG - AD}} \times$ ſeg. of the cir. AB whoſe height is $AB \times \frac{CG - AD}{CG} = \frac{\sqrt{4b^2 + \overline{D-d}|^2}}{D-d} \times : - ndd + FAE + \frac{dd}{DD} \times \frac{\frac{1}{2} \times \overline{D+d} - AD}{d - AD} \sqrt{\frac{D - AD}{d - AD}} \times$ ſegment of the circle AB whoſe height is $D \times \frac{d - AD}{d} = \frac{\sqrt{4b^2 + \overline{D-d}|^2}}{D-d} \times : - nd^2 + D^2 \times$ tab. ſeg. whoſe height is $\frac{AD}{D} + d^2 \times \frac{\frac{1}{2} \times \overline{D+d} - AD}{d - AD} \sqrt{\frac{D - AD}{d - AD}} \times$ tab. ſeg. whoſe height is $\frac{d - AD}{d}$.

COROLLARY IV.

If D coincide with A, the rules in the laſt cor. will become $\frac{VB}{BO} \times : AB^2 \times n - n \times IA \sqrt{AB \times CG} = \frac{n\sqrt{4b^2 + \overline{D-d}|^2}}{D-d} \times \overline{D^2 - \frac{D+d}{2}\sqrt{Dd}}$ for the convex ſurface of the ungula ABC.

And

PROBLEM XVI.

To find the convex ſurfaces of the elliptic ungulas of a cone, made by a plane cutting off a part of the baſe.

RULE.

Multiply continually together the four following quantities, viz. the quotient ariſing from the diviſion of

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And $\frac{VB}{BO} \times IA\sqrt{\overline{AB \times CG}} \times n = 2n \times VB \times IA\sqrt{\frac{CG}{AB}} = \frac{n\sqrt{4h^2+\overline{D-d}|^2}}{D-d} \times \frac{D+d}{2}\sqrt{Dd}$ for that of the oblique cone *ACV*.

Alſo $\frac{VB \times n}{BO} \times \overline{IA\sqrt{AB \times CG - CG^2}} = \frac{n\sqrt{4h^2+\overline{D-d}|^2}}{D-d} \times \frac{D+d}{2}\sqrt{Dd-d^2}$ for that of the complemental elliptic hoof *ACG*.

COROLLARY V.

If *CD* be parallel to *VA*, or the ſection a parabola; ſince its area (B) is $\frac{4}{3}CD \times DF = \frac{4}{3}CD\sqrt{\overline{AD \times DB}}$, the general theorem for the ungula will become $\frac{VB}{BO} \times \overline{\text{ſeg. } FBE - \frac{4}{3}ID\sqrt{AD \times DB}} = \frac{\sqrt{4h^2+\overline{D-d}|^2}}{D-d} \times$: ſeg. FBE (its height $BD = D-d$) $- \frac{2}{3} \times \overline{D-d}\sqrt{d \times \overline{D-d}}$ for the convex ſurface of the parabolic ungula *FEBC*.

And the rule in cor. 1 will be $\frac{VB}{BO} \times$: ſeg. $FAE + \frac{4}{3}ID\sqrt{\overline{AD \times DB}} = \frac{\sqrt{4h^2+\overline{D-d}|^2}}{D-d} \times$: ſeg. FAE (its height $AE = d$) $+ \frac{2}{3} \times \overline{D-d}\sqrt{d \times \overline{D-d}}$ for that of the part *AEFCG*.

Alſo that in cor. 2 will be $\frac{VB}{BO} \times$: ſeg. $FAE + \frac{4}{3}ID\sqrt{\overline{AD \times DB}} - AD^2 \times n = \frac{\sqrt{4h^2+\overline{D-d}|^2}}{D-d} \times$: ſegment FAE (its height $AD = d$) $+ \frac{2}{3} \times \overline{D-d}\sqrt{d \times \overline{D-d}} - ndd$, for that of the complemental parabolic ungula *FAECG*.

of the ſquare of the leſs diameter by that of the greater, the quotient ariſing from the diviſion of the difference between the part of the diameter of the baſe not cut off and half the ſum of the diameters by the difference between the ſaid part of the diameter of the baſe and the leſs diameter, the ſquare root of the quotient ariſing from the diviſion of the part cut off the diameter of the baſe by the difference between the part not cut off and the leſs diameter, and ſuch a ſegment of the baſe of the fruſtum whoſe height is equal to the product ariſing from the multiplica-

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COROLLARY VI.

If the $\angle CDB$ be greater than the $\angle VAB$, or the ſection be an hyperbola, its area being found and ſubſtituted for B in the general rules, will give the ſurfaces of the hyperbolic ungulas.

COROLLARY VII.

If the hyperbolic ſection be perpendicular to the baſe, DI will vaniſh, and the general theorems will become $\frac{VB}{BO} \times$ ſeg. $FBE = \frac{CB}{BI} \times$ ſeg. FBE (its height being BI) $= \frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d} \times$ ſeg. of the circle AB whoſe height is $\frac{D-d}{2}$ for the curve ſurface of the perpendicular ungula CIB.

Alſo $\frac{VB}{BO} \times$ ſeg. $FAE = \frac{CB}{BI} \times$ ſeg. whoſe height is $AI = \frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d} \times$ ſeg. of the cir. AB whoſe height is $\frac{D+d}{2}$ for that of the remaining part $AICV$.

And $\frac{VB}{BO} \times$ ſeg. $FAE -$ cir. $CG = \frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d} \times$: ſeg. of the circle AB whoſe height is $\frac{D+d}{2} - nd^2$ for that of the complemental perpendicular ungula $AICG$.

tion of the quotient of the greater diameter divided by the leſs into the difference between the leſs diameter and the part of the baſe diameter not cut off; and call the laſt product *R*.

Then

1. The difference between *R* and the baſe of the hoof multiplied by the quantity $\mathcal{Q}$, in the laſt problem, will produce the convex ſurface of the ungula required.

That is, $\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d} \times :$ ſeg. $FBE - \frac{dd}{DD} \times \frac{\frac{1}{2} \times \overline{D+d}-AD}{d-AD} \sqrt{\frac{BD}{d-AD}} \times$ ſeg. of the circle *AB* whoſe height is $D \times \frac{d-AD}{d} =$ the convex ſurface of the hoof *FEBC*. Uſing ſtill the ſame letters as in the former problems. (The fig. in page 179.

2. The ſum of *R* and the remaining ſegment of the baſe, not included by the hoof above, multiplied by $\mathcal{Q}$, will give the ſurface of the remaining part of the whole cone.

That is, $\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d} \times :$ ſeg. $FAE + \frac{dd}{DD} \times \frac{\frac{1}{2} \times \overline{D+d}-AD}{d-AD} \sqrt{\frac{BD}{d-AD}} \times$ ſeg. of the circle *AB* whoſe height is $D \times \frac{d-AD}{d} =$ the convex ſurface of the part *FEAVC*.

3. From the ſurface of the part in the laſt article ſubtract that of the little cone at the top, viz. the product of $\mathcal{Q}$ by the area of the top of the fruſtum, and the remainder will be the ſurface of the complemental ungula.

That is, $\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d} \times : -ndd +$ ſeg. $FAE + \frac{dd}{DD} \times \frac{\frac{1}{2} \times \overline{D+d}-AD}{d-AD} \sqrt{\frac{BD}{d-AD}} \times$ ſeg. of the cir. *AB* whoſe height

is

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is $D\times\frac{d-AD}{d}=$ the convex ſurface of the complemental hoof *FEAGC*.—All theſe are proved in cor. 3 prob. 15.

EXAMPLE.

Required the convex ſurfaces of the parts into which a cone is cut by a plane which enters the ſide at the diſtance of 18 inches from the baſe, and cuts off 20 inches of the diameter of the baſe, that diameter being 30 and the diameter at the top of the ſection $19\frac{1}{5}$ inches.

Here $D=30$, $d=19\frac{1}{5}$, $h=18$, $BD=20$, and $AD=10$. Whence

$Q=\frac{\sqrt{4h^2+\overline{D-d}|^2}}{D-d}=\frac{1}{5}\sqrt{109}=3{\cdot}48010217$.

The ſeg. $FBE=30^2\times{\cdot}55622573=500{\cdot}603157$.

The ſeg. $FAE=30^2\times{\cdot}22917243=206{\cdot}255187$.

$R=\overline{\frac{19\frac{1}{5}}{30}}\Big|^2\times\frac{14\frac{3}{5}}{9\frac{1}{5}}\sqrt{\frac{20}{9\frac{1}{5}}}\times$ ſeg. whoſe diameter is 30 and height $\frac{30\times9\frac{1}{5}}{19\frac{1}{5}}=\frac{8^2\times146\sqrt{46}}{5^3\times23^2}\times30^2\times$ tab. ſeg. whoſe height is $\frac{23}{48}=\frac{6^2\times8^2\times146\sqrt{46}}{5\times23^2}\times{\cdot}37187178=320{\cdot}761173$.

And $ndd={\cdot}785398\times\overline{19\frac{1}{5}}|^2=289{\cdot}529179$.

Wherefore

1. $Q\times\overline{FBE-R}=3{\cdot}48010217\times179{\cdot}841984=625{\cdot}868479=$ the convex ſurface of the hoof *FEBC*.

2. $Q\times\overline{FAE+R}=3{\cdot}48010217\times527{\cdot}01636=1834{\cdot}070779=$ that of the part *FAECV*.

3. $Q\times\overline{FAE+R-ndd}=3{\cdot}48010217\times237{\cdot}487181=826{\cdot}479654=$ that of the complemental hoof *FAECG*.

PRO-

PROBLEM XVII.

To find the convex ſurfaces of the parabolic hoofs made by a plane cutting a cone parallel to one ſide.

RULE.

Multiply $\frac{2}{3}$ of the difference of the diameters of the baſe and top of the fruſtum by the ſquare root of the product of the ſaid difference and the leſs diameter, and call the product S.

Then

1. The difference between S and the baſe of the hoof multiplied into $\mathcal{Q}$, in the 15th problem, will give the ſurface of the hoof.

That is, $\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d} \times :$ ſegment $FBE - \frac{2}{3} \times \overline{D-d} \sqrt{d \times \overline{D-d}} =$ the convex ſurface of the ungula $FEBC$. Uſing ſtill the ſame letters.

2. The Sum of S and the complemental baſe multiplied by $\mathcal{Q}$, will produce the convex ſurface of the remaining part of the cone.

That is, $\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d} \times :$ ſegment $FAE + \frac{2}{3} \times \overline{D-d} \sqrt{d \times \overline{D-d}} =$ the ſurface of the part $FAECV$.

3. The difference between the circle CG and the ſum of S and the complemental baſe multiplied by $\mathcal{Q}$, will give the ſurface of the complemental hoof.

That is, $\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d} \times :$ ſegment $FAE + {}^{2} \times \overline{D-d} \sqrt{d \times \overline{D-d}} - ndd =$ the convex ſurface of the complemental hoof $FAECG$.

Ex-

EXAMPLE.

Required the convex ſurfaces of the parabolic hoofs of the fruſtum of a cone whoſe height is 18 and its diameters 30 and $19\frac{1}{5}$ inches.

Here again $Q = 3{\cdot}48010217$.

$S = \frac{1}{3} \times \overline{D-d}\sqrt{d \times \overline{D-d}} = \frac{1}{3} \times 10\frac{4}{5}\sqrt{19\frac{1}{5} \times 10\frac{4}{5}} = \frac{2\times54\times6\times4\times3}{3\times5\times5}$ $= \frac{2592}{25} = 103{\cdot}68$.

The ſeg. $FBE = 30^2 \times$ tab. ſeg. whoſe verſed ſine is $\left(\frac{10\frac{4}{5}}{30} = \frac{3{\cdot}6}{10} =\right) {\cdot}36 = 900 \times {\cdot}25455055 = 229{\cdot}095495$.

The ſeg. $FAE = 30^2 \times$ tab. ſeg. whoſe height is $\left(\frac{19\frac{1}{5}}{30} = \frac{6{\cdot}4}{10} =\right) {\cdot}64 = 900 \times \overline{{\cdot}785398 - {\cdot}25455055} = 900 \times {\cdot}53084761 = 477{\cdot}762849$.

And ndd, as before, $= 289{\cdot}529179$.

Whence

1. $Q \times \overline{FBE - S} = 3{\cdot}48010217 \times 125{\cdot}415495 = 436{\cdot}458737 =$ the convex ſurface of the hoof *FEBC*.

2. $Q \times \overline{FAE + S} = 3{\cdot}48010217 \times 581{\cdot}442849 = 2023{\cdot}480521 =$ that of the part *FAECV*.

3. $Q \times \overline{FAE + S - ndd} = 3{\cdot}48010217 \times 291{\cdot}91367 = 1015{\cdot}889396 =$ that of the complemental hoof *FAECG*.

PROBLEM XVIII.

To find the curve ſurfaces of the hyperbolic hoofs of the fruſtum of a cone, made by a plane cutting it perpendicular to the baſe.

RULE.

1. Multiply the quantity Q, found as in the preceding problems, by the baſe of the hoof, and the product will be its ſurface.

That is, $\frac{VB}{BO}\times$ ſeg. $FBE=\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d}\times$ ſeg. of the circle AB whoſe height is $\frac{D-d}{2}=$ the curve ſurface of the perpendicular hoof $FEBC$. (See the figure at page 175.)

2. Multiply $\mathcal{Q}$ by the complemental baſe, and the product will be that of the complemental part of the whole cone.

That is, $\frac{VB}{BO}\times$ ſeg. $FAE=\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d}\times$ ſeg. of the circle AB whoſe height is $\frac{D+d}{2}(=AD=AI)=$ the curve ſurface of the part $FAECV$.

3. Multiply $\mathcal{Q}$ by the difference between the complemental ſegment and the circle CG, and the product will be that of the complemental hoof.

That is, $\frac{VB}{BO}\times\overline{\text{ſeg. }FAE-\text{cir. }CG}=\frac{\sqrt{4b^2+\overline{D-d}|^2}}{D-d}\times:$ $-ndd+$ ſeg. of the circle AB whoſe height is $\frac{D+d}{2}=$ the curve ſurface of the complemental hoof $FAECG$. Uſing ſtill the ſame letters as in the former problems.

Note. The two firſt articles are the ſame rule, and ſerve for the whole cone as well as any perpendicular part.

EXAMPLE.

Required the curve ſurfaces of the perpendicular hoofs of a cone whoſe baſe diameter is 30 inches; the height of the ſection being 18 and the diameter of the cone at the top of the ſection $19\frac{1}{2}$ inches.

Here again $\begin{cases}\mathcal{Q}=3{\cdot}48010217, \\ ndd=289{\cdot}529179.\end{cases}$

The ſegment $FBE = 30^2 \times$ tab. ſeg. whoſe height is $\left(\frac{30-19\frac{1}{5}}{2\times30}\right) \cdot 18 = 900 \times \cdot 09613453 = 86 \cdot 521077$.

The ſegment $FAE = 30^2 \times$ tab. ſeg. whoſe height is $\left(\frac{30+19\frac{1}{5}}{2\times30}\right) \cdot 82 = 900 \times \cdot 68926363 = 620 \cdot 337267$.

Whence

1. $Q \times FBE = 3 \cdot 48010217 \times 86 \cdot 521077 = 301 \cdot 102188$ = the curve ſurface of the hoof *FEBC*.

2. $Q \times FAE = 3 \cdot 48010217 \times 620 \cdot 337267 = 2158 \cdot 837068$ = that of the part *FEACV*.

3. $Q \times \overline{FAE - ndd} = 3 \cdot 48010217 \times 330 \cdot 808088 = 1151 \cdot 245945$ = that of the complemental hoof *FEACG*.

PROBLEM XIX.

To find the curve ſurface of a ſphere, or of any ſegment or zone of it.

RULE.

Multiply the circumference of the ſphere into the height of the part required, and the product will be the curve ſurface, whether it be ſegment, zone, hemiſphere, or the whole ſphere.*

Note. The height of the whole ſphere is its diameter.

Ex-

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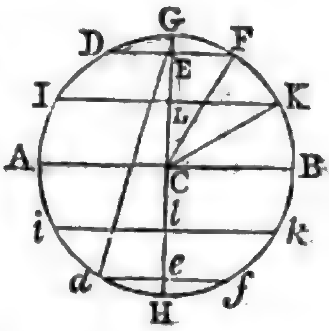
* DEMONSTRATION.

Put $d =$ the diameter AB or $2CG$, $x = GE$ the height of the zone $ADFB$, $y = ED$, $z = DA$, $p = 3 \cdot 14159$ &c. and $s =$ the ſurface required.

Then $\dot{s} = 2py\dot{z}$; but as $y : \frac{1}{2}d :: \dot{x} : \dot{z} = \frac{d\dot{x}}{2y}$; therefore $\dot{s} = pd\dot{x}$, and conſequently $s = pdx$; viz. the product of the circumference of the ſphere and height of the zone, for pd is = the circumference of the circle whoſe diameter is d.



Co-

EXAMPLE I.

If the diameter or axe of the earth be $7957\frac{3}{4}$ miles, what is the whole ſurface, ſuppoſing it a perfect ſphere?

$7957\frac{3}{4} \times 3{\cdot}141592 = 25000$ miles, (extremely near) = the circumference.

Then $7957\frac{3}{4} \times 25000 = 198943750$ ſquare miles = the whole ſurface required. And the half is 99471875 = the ſurface of the hemiſphere.

Ex-

COROLLARY I.

When x becomes $= CG = \frac{1}{2}d$, s will be $= \frac{1}{2}pdd$ = the ſurface of the hemiſphere. And conſequently that of the whole ſphere is pdd, the product of the circumference and height or diameter.

COROLLARY II.

To or from $\frac{1}{2}pdd$, the ſurface of the hemiſphere, add or ſubtract pdx, that of the zone $ADFB$, and the remainder $pd \times \overline{\frac{1}{2}d \pm x} = pd \times GE$ will be the ſurface of the ſegment DGF, viz. the product of the circumference and height. So that the rule is general. *Q.E.D.*

COROLLARY III.

The ſurfaces of ſpheres, and alſo of their ſimilar parts, are to each other as the ſquares of their diameters. For, by exterminating the common given quantity p, they are as d, the diameter, into the height of the part; but the heights of ſimilar parts are as the diameters; wherefore &c.

COROLLARY IV.

The ſurfaces of any ſegments or zones of a ſphere are to each other, or to that of the whole ſphere, as their heights. For pd is common to them all.

COROLLARY V.

Or the ſurface of any ſegment or zone of a ſphere is as its height.

COROLLARY VI.

The ſurface of any ſegment or zone is equal to 4 times a circle whoſe diameter is a mean proportional between its height and the diameter of the ſphere. For this circle is $= \frac{1}{4}p \times \overline{\sqrt{dx}}|^2 = \frac{1}{4}pdx$.

Co-

EXAMPLE II.

To find the ſurface of the two frigid zones of the earth.

Note. The frigid zones are the two oppoſite ſegments DGF, dHf, in which each of the arcs DG, dH, or half the breadth of the zone, is $23\frac{1}{2}$ degrees.

Draw the radius CF; then the angle $FCE = 23\frac{1}{2}$ degrees, and the angle $EFC = 90 - 23\frac{1}{2} = 66\frac{1}{2}$ degrees; alſo the angle $E = 90$ degrees.

Then, as $s.\angle E : s.\angle F :: FC = 3978\frac{7}{8} : CE = 3648\cdot8675054$.

And $EG = GC - CE = 3978\cdot875 - 3648\cdot8675054 = 330\cdot0074946 =$ the height.

Hence 25000 (the circumference) $\times\ 330\cdot0074946 = 8250187\cdot365 =$ the ſurface of the ſegment DGF or frigid zone.

COROLLARY VII.

Hence the ſurface of a ſegment DGF is equal to 4 times the circle whoſe diameter is the chord GD drawn from the vertex to the extremity of the baſe; or equal to a circle whoſe radius is that chord. For $GD = \sqrt{HG \times GE} = \sqrt{dx}$.

COROLLARY VIII.

And hence the ſurface of a ſphere is equal to 4 times the area of a great circle of it; that is, 4 times the area of a circle of the ſame diameter with the ſphere. And conſequently the ſurface of an hemiſphere is double the area of its baſe.

COROLLARY IX.

Or the ſurface of a ſphere is equal to a circle whoſe diameter is double to that of the ſphere.

COROLLARY X.

The ſurface of any ſegment or zone of a ſphere is equal to the curve ſurface of a cylinder of the ſame height with it, and whoſe diameter is equal to that of the ſphere.

COROLLARY XI.

Hence the ſurface of a whole ſphere, or of a hemiſphere, is equal to the curve ſurface of the circumſcribed cylinder.

EXAMPLE III.

To find the ſurface of the torrid zone *IKki*, which extends to the diſtance of (*IA* or *Ai*) $23\frac{1}{2}$ degrees on each ſide from *AB* the diameter.

Here the $\angle LKC=23\frac{1}{2}$ degrees.

Then, $s.\ \angle L : s.\ \angle LKC :: KC : CL = 1586{\cdot}572$.

And $Ll=2CL=3173{\cdot}144$ = the height.

Wherefore $25000\times 3173{\cdot}144 = 79328600$ ſquare miles = the ſurface of the torrid zone.

EXAMPLE IV.

Required the convex ſurface of each of the temperate zones *DFKI*, *dfki*, which are included between the frigid and torrid zones.

By the 1ſt ex. $CE=3648{\cdot}8675$
and by the 2d ex. $CL=1586{\cdot}572$
their difference is $LE=2062{\cdot}2955$ = the height.

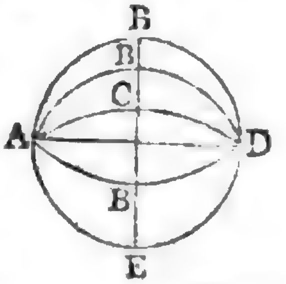
Wherefore $25000\times 2062{\cdot}2955=51557387\frac{1}{2}$ ſquare miles = each temperate zone.

Hence the two frigid zones = 16500375
the two temperate zones = 103114775
the torrid zone = 79328600
their ſum is 198943750 = the ſurface of the ſphere.

PROBLEM XX.

To find the lunar ſurface ABDCA included between two great circles ABD, DCA of a ſphere.

RULE I.



Multiply the diameter *AD* into the breadth *BC* of the ſurface in the middle, that is, into the arc that meaſures the angle *BAC* of inclination of the two circles, and the product will be the ſurface *ABDCA*.

That is, if $d = AD$ the diameter,

$a =$ the length of the arc BC over the middle of the ſurface.

Then ad is the ſurface.

[*] RULE II.

As four right angles are to the ſurface of the ſphere;
Or, As one right angle is to a great circle of the ſphere;
So is the angle made by the two great circles,
To the ſurface included by them.

EXAMPLE.

Required the ſurface included by two great circles forming an angle of 25 degrees, the diameter of the ſphere being 10 feet.

1. By rule 1.

Firſt, $3\cdot1415926 \times 10 = 31\cdot415926 =$ the circumference.

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[*] DEMONSTRATION.

If the great circle BCE, whoſe poles are A and D, be conceived to be divided into an indefinite number of equal parts, and great circles be conceived to be drawn through the points of diviſion and through A and D; it is evident that the whole ſurface will be divided, by the circles, into the ſame number of parts ſimilar and equal to one another, and that, therefore, the ſurface included between any two of thoſe circles will be as the number of parts, or as the arc of the circle BE included by them; wherefore as the whole circumference BE is to the arc BC, or as four right angles to the angle BAC, ſo is the ſurface of the ſphere to the lunar ſurface $ABDCA$; or alſo as one right angle is to the angle BAC, ſo is the area of a great circle of the ſphere ($\frac{1}{4}$ of the ſurface) to the lunar ſurface. Which is rule 2.

COROLLARY.

If d be the diameter and c the circumference of the ſphere, and a the arc BC; then, from the proceſs above, $c : a :: cd$ (the ſurface of the ſphere) $: ad =$ the lunar ſurface. Which is rule 1.

And 360:25(::72:5):: 31·415926:2·181661 = the greateſt breadth of the ſurface.

Then 2·181661×10 = 21·81661 = the ſurface required.

2. By rule 2.

Here 31·415926 (the circumference) ×10 = 314·15926 = the ſurface of the ſphere.

And 360 : 25 :: (72: 5::) 314·15926 : 21·81661 = the ſurface required.

PROBLEM XXI.

To find the area of a ſpheric triangle, that is, the ſpheric ſurface included by the arcs of three great circles of the ſphere interſecting one another.

RULE.

As 8 right angles or 720°,
To the ſurface of the ſphere;
Or, As 2 right angles or 180°,
To a great circle of the ſphere;
So is the exceſs of the 3 angles above 2 right angles,
To the area of the triangle.*

That

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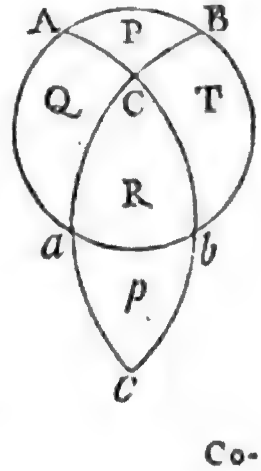
* DEMONSTRATION.

For having produced all the ſides of the triangle *ABC* (*P*) till they interſect again and form the triangle *p*, which by the principles of the ſphere will be equal to the former triangle *P*; put $\frac{1}{2}S = \frac{1}{2}$ the ſurface of the ſphere $= P + Q + R + T$. Then, by the laſt problem,

$$\begin{cases} 180^\circ : A :: \frac{1}{2}S : P+T, \\ 180 \ : B :: \frac{1}{2}S : P+Q, \\ 180 \ : C :: \frac{1}{2}S : P+R \ (=p+R), \end{cases}$$

hence $180 : A+B+C :: \frac{1}{2}S : (3P+T+Q+R=)\ 2P+\frac{1}{2}S,$

and as $180 : A+B+C-180 :: \frac{1}{2}S : 2P :: \frac{1}{4}S : P = S \times \frac{A+B+C-180}{720}$. *Q.E.D.*



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That is, $pdd \times \frac{s-180}{720}$ = the area of the triangle, putting $p = 3{\cdot}14159$ &c. d = the diameter of the ſphere, and s = the ſum of the 3 angles of the triangle.

EXAMPLE.

If the angles be 55, 60, and 85 degrees, what is the trilineal ſurface; ſuppoſing the diameter to be 10?

Here $\cdot 7853982 \times 10 \times 10 = 78{\cdot}53982 = \frac{1}{4}$ of the ſurface of the ſphere.

Then $180 : 20\ (= 55+60+85-180) :: 78{\cdot}53982 : \frac{78{\cdot}53982}{9} = 8{\cdot}72664$ = the area of the triangle required.

PROBLEM XXII.

To find the area of a ſpheric polygon, or to find the ſpherical ſurface included by any number of interſecting great circles.

RULE.

As 8 right angles or 720°,
To the ſurface of the ſphere;
Or, As 2 right angles or 180°,
To a great circle of the ſphere;
So is the exceſs of all the angles above the product of 180 and 2 leſs than the number of angles,
To the area of the ſpheric polygon.

That

COROLLARY.

When $P = 0$, then ($A+B+C$ or) $s = 180$; but when $P = \frac{1}{2}pdd$, half the ſurface of the ſphere, then $s - 180 = 360$, or $s = 540$; conſequently $A+B+C$ is always between 180 and 540, that is, greater than 2 and leſs than 6 right angles.

That is, putting $n =$ the number of angles,
$s =$ the ſum of all the angles,
$d =$ the diameter of the ſphere,
and $p = 3{\cdot}14159$ &c.

Then $pd^2 \times \frac{s - 180 \times \overline{n-2}}{720} =$ the area of the ſpheric polygon.*

EXAMPLE.

What ſurface is included by the intercepted arcs of five interſecting great circles of a ſphere of 10 feet diameter; ſuppoſing the ſum of the angles formed by thoſe arcs to be 640 degrees.

Here the ſurface of the ſphere is $3{\cdot}1415926 \times 10 \times 10 = 314{\cdot}15926$.

And

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* DEMONSTRATION.

For if the polygon be ſuppoſed to be divided in as many triangles as it has ſides by great circles drawn to all the angles through any point within it, forming at that point the vertical angles of all the triangles. Then, by the laſt problem, it will be in any one triangle,

As $720 : S ::$ ſum of its angles $- 180 :$ its area,

Wherefore by compoſition, As $720 : S :: s - 180n :$ ſum of all the triangles or area of the polygon.

But all the vertical angles $= 360$ or 180×2.

Wherefore, As $720 : S :: s - 180 \times \overline{n-2} : S \times \frac{s - 180 \times \overline{n-2}}{720} =$ the area of the polygon. *Q.E.D.*

COROLLARY.

When the polygon is $= 0$, then s is $= 180 \times \overline{n-2}$; and when the polygon is $=$ the ſemi-ſpheric ſurface, then $s = 180 \times \overline{n-2} + 360 = 180 \times n$; conſequently s the ſum of all the angles of any polygon is always between $180 \times \overline{n-2}$ and $180 \times n$, that is, leſs than n times 2 right angles, but greater than $\overline{n-2}$ times 2 right angles, n being the whole number of angles.

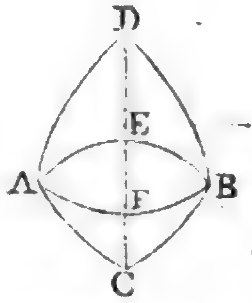
And $180\times\overline{n-2}=180\times3=540$, which taken from 640 leaves 100.

Whence, as $720:100::314{\cdot}15926:\frac{392{\cdot}69908}{9}=43{\cdot}63323=$ the area of the polygon required.

PROBLEM XXIII.

To find the ſurface AEBF *included between two interſecting little circles of a ſphere.*

RULE.



Through the poles *D*, *C*, and interſections *A*, *B*, of the two little circles *AFB*, *AEB*, draw the great circles *AD*, *DB*, *BC*, *CA*; and alſo the great circle *DEFC*.

By prob. 21 find the area of the triangle *BCD*, having firſt found the angles at *B*, *C*, and *D*, from the given ſides, by the principles of ſpherical trigonometry.

By prob. 19 find the ſurface of the ſegment cut off by the circle of which *AFB* is a part, thus, viz. As the diameter is to the verſed ſine of the arc *DF* or *DB*, ſo is the ſurface of the ſphere to that of the ſegment.

Then as 4 right angles, to the $\angle BDC$, ſo is that ſurface, to the part of it *DBF*.

In the ſame manner find the ſurface *CBE*.

Then from the ſum of *DBF* and *CBE* take the triangle *DBC*, and the remainder will be the part *BEF*; the double of which will be the whole *AEBFA*.

PROBLEM XXIV.

To find the ſolidity of a ſphere or globe.

*RULE I.

Multiply the ſurface by $\frac{1}{3}$ of the radius or by $\frac{1}{6}$ of the diameter, and the product will be the ſolidity.

RULE II.

Multiply the cube of the diameter by ·5236, and the product wil be the ſolidity.

That is $\left(\frac{3\cdot1416}{6}\right)\cdot5236d^3 =$ the ſolidity.

Ex-

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*DEMONSTRATION.

The ſphere may be conſidered as conſtituted of an infinite number of pyramids whoſe baſes compoſe the ſpheric ſurface, and all whoſe vertexes concur in the center, their common height being equal to the radius of the ſphere. And conſequently the ſphere, or any ſpheric pyramid (being a part contained within right lines drawn from the ſurface to the center) is equal to a pyramid whoſe baſe is equal to the ſpheric ſurface and height equal to the radius. And therefore the ſurface of the whole, or of any ſuch part, being drawn into $\frac{1}{3}$ of the radius will give the ſolidity, as in rule 1.

COROLLARY I.

Since the ſurface of the ſphere is $= pdd$, we ſhall have $pdd \times$ ($\frac{1}{3}$ of the radius or) $\frac{1}{6}d = \frac{1}{6}pd^3 = \cdot523598775$ &c. $\times d^3$ (which is rule 2.) $= \frac{2}{3}$ of a cylinder of the ſame diameter and height.

COROLLARY II.

If h be the height or verſed ſine of any ſegment; then, ſince $pdh =$ its ſurface, $\frac{1}{6}pd^2h = \cdot523598775$ &c. $\times d^2h$ will be the ſolidity of the ſpheric pyramid or cone whoſe baſe is the ſurface of the ſegment.

COROLLARY III.

Hence ſpheres and their ſimilar pyramids, and alſo any other ſimilar parts of them, are as the cubes of the diameters.

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EXAMPLE.

Suppoſing the earth to be ſpherical, and its diameter $7957\frac{3}{4}$ miles, what is its ſolidity.

1. By rule 1.

By exam. 1 of prob. 19 the ſurface is 198943750, then $198943750\times\frac{7957\frac{3}{4}}{6}=263857437760$ miles = the ſolidity.

2. By the 2d rule.

Here $\cdot5236d^3=\cdot5236\times\overline{7957\frac{3}{4}}|^3=263857624944$ miles = the ſolidity by this rule: the difference ariſing by taking the number ·5236 a little too great.

COROLLARY IV.

If to or from ($\frac{1}{6}pd^2h$) the ſpheric cone be added or ſubtracted ($\frac{1}{4}p\times\frac{\mp\frac{1}{2}d\pm h}{3}\times4h\times\overline{d-h}=\mp\frac{1}{6}pd^2h\pm\frac{1}{2}pdh^2\mp\frac{1}{3}ph^3$) the cone whoſe baſe is the ſame with the baſe of the ſegment, and whoſe vertex is in the center, the ſum or difference $\frac{1}{2}pdh^2-\frac{1}{3}ph^3=\frac{1}{3}ph^2\times\overline{\frac{3}{2}d-h}$, will be the ſpheric ſegment whoſe height is h, either greater or leſs than the hemiſphere, or of whatever magnitude h is, not exceeding d. Or if r be = the radius of the ſegment's baſe, ſince $d=\frac{r^2+h^2}{h}$, the ſegment will be $\frac{1}{2}pr^2h+\frac{1}{6}ph^3=\frac{1}{6}ph\times\overline{3r^2+h^2}$.

COROLLARY V.

Hence the difference between two ſegments whoſe heights are H, h, and the radiuſes of their baſes R, r, will give for the fruſtum or zone $\frac{1}{6}p\times:3R^2H+H^3-3r^2h-h^3$, which, putting a for the altitude of the fruſtum, and exterminating H and h by means of the two equations $\frac{R^2+H^2}{H}=\frac{r^2+h^2}{h}$, and $a=H-h$, will become $\frac{1}{2}ap\times\overline{R^2+r^2+\frac{1}{3}a^2}$.

COROLLARY VI.

If one end of the fruſtum paſs through the center, then $R^2=\frac{1}{4}d^2=r^2+a^2$, and the laſt theorem will become $ap\times\overline{r^2+\frac{2}{3}a^2}=ap\times\overline{\frac{1}{4}d^2-\frac{1}{3}a^2}$.

PROBLEM XXV.

To find the ſolidity of the ſegment of a ſphere.

RULE I.

To three times the ſquare of the radius of its baſe add the ſquare of its height, multiply the ſum by the height and the product by ·5236; and the laſt product will be the ſolidity.

That is, if $r=DE$ the radius of its baſe,
$h=EG$ the height;

Then $\cdot5236h\times\overline{3rr+hh}=$ the ſolidity of the ſegment *DGF*. By cor. 4 to the laſt problem. [See the figure in page 1[illegible].]

RULE II.

From three times the diameter of the ſphere ſubtract twice the height of the fruſtum, multiply the difference into the ſquare of the height and into ·5236, and the laſt product will be the ſolidity.

That is, if $d=GH$ the diameter of the ſphere,
$h=EG$ the height of the fruſtum;

Then $\cdot5236h^2\times\overline{3d-2h}=$ the ſolidity of *DGF*. By cor. 4 to the laſt problem.

EXAMPLE.

What is the ſolidity of each of the frigid zones of the earth, the axe being $7957\frac{1}{4}$ miles, and half the

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COROLLARY VII.

Hence the middle zone, or the double of the laſt expreſſion will be $2ap\times\overline{r^2+\frac{2}{3}a^2}=2ap\times\overline{\frac{1}{4}d^2-\frac{1}{3}a^2}$, where a is $\frac{1}{2}$ its altitude and $r=\frac{1}{2}$ the diameter of each end. But if A be its whole altitude, and D the diameter of each end, thoſe theorems will become $\frac{1}{4}Ap\times\overline{D^2+\frac{2}{3}A^2}=\frac{1}{4}Ap\times\overline{d^2-\frac{1}{3}A^2}$.

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the breadth, or arc *VA*, of the zone being 23½ degrees?

By rule 2.

As 1 = tabular radius : 3978⅞ = radius of the earth :: ·0829399 = tab. verſed ſine of 23½ degrees : ·0829399 × 3978⅞ = 330·0074946 = the verſed ſine or height of the ſegment.

Then $\cdot 5236h^2 \times \overline{3d-2h}$ = ·5236 × 330·0074946 × 330·0074946 × 23213·2350108 = 1323679710 = the content of the ſegment.

By rule 1.

As 1 : 3978⅞ :: ·3987491 = tabular ſine of 23½ degrees : ·3987491 × 3978⅞ = 1586·57282526 = the radius of the ſegment's baſe.

Then $\cdot 5236h \times \overline{3r^2+h^2}$ = ·5236 × 330·0074946 × 7660544·936 = 1323680299·69 = the ſolidity of the ſegment.

PROBLEM XXVI.

To find the ſolidity of a fruſtum or zone of a ſphere.

RULE.

Add together the ſquares of the radiuſes of the ends and ⅓ of the ſquare of their diſtance or of the height, multiply the ſum into the ſaid height, and the product multiplied into 1·5708 will produce the content required.

That is, $\overline{R^2+r^2+\frac{1}{3}h^2} \times \frac{1}{2}ph$ = the ſolidity of the fruſtum whoſe height is *h*, and the radiuſes of its ends *R* and *r*, *p* being equal 3·1416. By cor. 5 to prob. 24.

EXAMPLE I.

What is the ſolidity of the fruſtum of a ſphere, the diameter of whoſe greater end is 4 feet, the diameter of the leſs 3 feet, and the height $2\frac{1}{2}$ feet.

Here $\overline{R^2+r^2+\frac{1}{3}h^2}\times1\cdot5708h=\overline{2^2+1\cdot5^2+\frac{1}{3}\times2\cdot5^2}\times1\cdot5708\times2\frac{1}{2}=8\frac{1}{3}\times3\cdot927=32\cdot725$ = the ſolidity of the fruſtum required.

EXAMPLE II.

What is the ſolidity of each temperate zone of the earth, they extending from $23\frac{1}{2}$ degrees to $66\frac{1}{2}$ degrees of latitude, and the diameter of the earth being $7957\frac{3}{4}$ miles.

By the example to the laſt problem the radius of the top is $1586\cdot57282526$.

And as $1:3978\frac{7}{8}::\cdot9170601$ = tabular ſine of $66\frac{1}{2}$ degrees : $\cdot3978\frac{7}{8}\times\cdot9170601=3648\cdot86750538$ = the radius of the baſe.

Alſo by example 4 prob. 19, the height is $2062\cdot2955$.

Then $\overline{R^2+r^2+\frac{1}{3}h^2}\times1\cdot5708h=17249136\times2062\cdot2955$ $1\cdot5708=55877778668$ = the ſolidity of each temperate zone.

Otherwiſe.

Since the radiuſes of the ends of this zone are the ſines of $23\frac{1}{2}$ and $66\frac{1}{2}$ degrees, which are complements the one of the other, the ſum of the ſquares of thoſe radiuſes will be equal to the ſquare of the radius of the ſphere; and therefore $\overline{\frac{1}{4}dd+\frac{1}{3}hh}\times1\cdot5708h=17249136\times2062\cdot2955\times1\cdot5708=55877778668$ = the content the ſame as before.

PRO-

PROBLEM XXVII.

To find the ſolidity of the middle zone of a ſphere.

RULE.

Multiply either the ſum of the ſquare of the diameter of the end and $\frac{2}{3}$ of the ſquare of the height, or the difference between the ſquare of the diameter of the ſphere and $\frac{1}{3}$ of the ſquare of the height, into the height, and the product by ·7854 for the content.

That is, $\overline{dd+\frac{2}{3}hh}\times{\cdot}7854h$ or $\overline{DD-\frac{1}{3}hh}\times{\cdot}7854h=$ the content of the middle zone whoſe height is h, the diameter of each end d, and the diameter of the ſphere D. By cor. 6. to prob. 24.

EXAMPLE I.

Required the ſolidity of the middle zone of a ſphere whoſe top and bottom diameters are each 3 feet, and height 4 feet.

Here $\overline{dd+\frac{2}{3}hh}\times{\cdot}7854h=\overline{3^2+\frac{2}{3}\times4^2}\times4\times{\cdot}7854=59\times\times4\times{\cdot}2618=61{\cdot}7848=$ the content required.

EXAMPLE II.

What is the ſolidity of the torrid zone of the earth, which extends to 23½ degrees on each ſide of the equator; the diameter of the earth being 7957¾ miles?

By the example to prob. 25, the ſine of 23½ degrees, or ½ the height of the zone, is 1586·57282526, and the whole height is $3173{\cdot}14565052=h$.

Then $\overline{DD-\frac{1}{3}hh}\times{\cdot}7854h=$
$\overline{7957{\cdot}75^2-\frac{1}{3}\times3173{\cdot}14565052^2}\times3173{\cdot}14565052\times{\cdot}7854=149455081137=$ the content required.

SCHOLIUM.

From the laſt three problems we find that
The two frigid zones = 2647359420
The two temperate zones = 111755557336
The torrid zone = 149455081137
whoſe ſum 263857997893 is the whole ſphere, nearly the ſame as found in prob. 24.

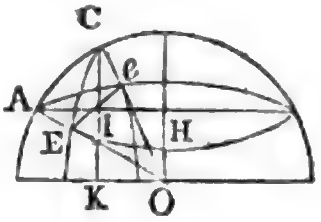
PROBLEM XXVIII.

To find the ſolidity of the ſecond ſegment of a ſphere.

DEFINITION.

A ſecond ſegment is a part cut off a ſegment by a plane perpendicular to the baſe.

RULE.



By prob. 23 find the curve ſurface of the ſecond ſegment *AECe*, which being drawn into ⅓ of the radius of the ſphere, will produce the content of the ſpheric ſector *OEAeC*; from which if there be taken the pyramid *OEAe* whoſe baſe is the ſegment *AEe* and height *OH*, and from the remainder be taken the pyramid *OECe* whoſe baſe is the ſegment *CEe* and height *KO* or *HI*, it is evident that the laſt remainder will be the ſecond ſegment *AEeC*: for theſe two pyramids and the ſecond ſegment compoſe the ſpheric pyramid.

PRO-

PROBLEM XXIX.

To find the ſurface of a circular ſpindle, or of any ſegment or fruſtum of it.

DEFINITION.

A circular ſpindle is a ſolid *ACBDA* conceived to be generated by the rotation of a ſegment *ACB* of a circle about its chord *AB* as an axe.—The diſtance *FE* of the middle of the chord, or center of the ſpindle, from the center of the circle, or of the revolving arc, is called the central diſtance.

RULE.

From the product of the height of the ſolid and radius of the revolving arc ſubtract the product of the ſaid arc and central diſtance, multiply the remainder by 3·1416, and double the product will be the ſurface deſcribed by that arc, whether it be the whole or any part of the ſpindle.*

That

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* DEMONSTRATION.

Put $z =$ the arc CP, $x =$ its ſine RE, r and c for the radius and central diſtance, and $p = 3{\cdot}14159$ &c. Then the fluxion of the ſurface $\dot{s}$ is $= 2p\dot{z} \times RP = 2p \times \overline{\dot{z}\sqrt{rr-xx}-c\dot{z}}$; but, by the property of the circle, $\dot{z}\sqrt{rr-xx}$ is $= r\dot{x}$; therefore $\dot{s} = 2p \times \overline{r\dot{x}-c\dot{z}}$, and $s = 2p \times \overline{rx-cz} = 2p \times \overline{r \times ER - c \times CP}$.

COROLLARY I.

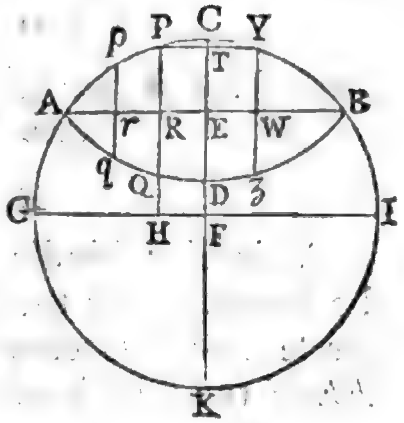
When $ER = EA$, the rule becomes $2p \times r \times \overline{EA - c \times CA}$ for the ſurface of half the ſpindle, or $2p \times r \times \overline{AB - c \times ACB}$ for that of the whole.

COROLLARY II.

If from the ſurface of the ſemi-ſpindle be taken that of the fruſtum, there will remain $2p \times \overline{r \times RA} - c \times AP$ for that of the ſegment PAQ: ſo that the rule is general.

Co-

That is, if
$r = FC$ the rad. of the arc,
$c = FE$ the central diſtance,
$h = Rr$ the height of any part $pPQq$,
$a = Pp$ its revolving arc,
& $p = 3{\cdot}1416$.
Then $\overline{hr - ac} \times 2p =$ the ſurface of that part $pPQq$.



EXAMPLE.

Required the ſurface of a circular ſpindle whoſe greateſt diameter is 30 and its length 40 inches.

Here, by the property of the circle, $EK = AE^2 \div EC = 20^2 \div 15 = 400 \div 15 = 26\frac{2}{3}$, and $\overline{26\frac{2}{3} + 15} \div 2 = 41\frac{2}{3} \div 2 = 20\frac{5}{6} =$ the radius of the circle.

And $EF = FC - CE = 20\frac{5}{6} - 15 = 5\frac{5}{6} =$ the central diſtance.

Alſo,

COROLLARY III.

When E coincides with F, c vaniſhes, the ſpindle becomes a ſphere, and the theorem becomes barely $2prx$, the ſame with that before found for the ſphere.

COROLLARY IV.

From cor. 1, it appears that the radius is to the coſine of an arc always in a greater proportion than that of the arc to its ſine, or of the double arc to its chord; but thoſe ratios approximate to an equality as the arc diminiſhes, till, when the arc vaniſhes, they become accurately equal, and the arc and chord vaniſh alſo in a ratio of equality.—Conſequently, in ſmall arcs the arc is a fourth proportional to the coſine, ſine, and radius, nearly. Thus, the ſine and coſine of an arc of 1 degree are $\cdot 0174524$ and $\cdot 9998477$ to the radius 1; then $\cdot 9998477 : \cdot 0174524 :: 1 : \cdot 0174550 =$ the arc nearly, the true figures being $\cdot 0174533$.

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Alſo, as $20\frac{5}{6} : 20 :: 1 : 24 \div 25 = 96 \div 100 = {}^{\cdot}96 =$ the tabular ſine of the arc *AC*, to which belong $73^{\cdot}7398$ degrees, the double of which $147^{\cdot}4796$ are the number of degrees in the whole arc *ACB*. And therefore by rule 1 prob. 6 ſect. 1 part 2, we have ${}^{\cdot}01745329 \times 147^{\cdot}4796 \times 20\frac{5}{6} = 53^{\cdot}62508 =$ the length of the revolving arc.

Wherefore, by the rule, $\overline{br - ac} \times 2p = \overline{20\frac{5}{6} \times 40 - 53^{\cdot}62508 \times 5\frac{5}{6}} \times 6^{\cdot}2832 = \overline{25 \times 40 - 53^{\cdot}62508 \times 7} \times \frac{5}{6} \times 6^{\cdot}2832 = 624^{\cdot}62444 \times 5^{\cdot}236 = 3270^{\cdot}5335 =$ the ſurface required.

PROBLEM XXX.

To find the ſolidity of a circular ſpindle.

RULE.

From $\frac{1}{3}$ of the cube of half the length of the ſpindle ſubtract the product of the central diſtance and half the generating circular ſegment, multiply the remainder by four times $3^{\cdot}14159$ &c. and the product will be the content of the ſpindle.*

That

---

* GENERAL INVESTIGATION.

Putting $RE = x$ (ſee the fig. to the laſt prob.) $EF = c$, $3^{\cdot}14159$ &c. $= p$; the fluxion of the ſolid $\dot{S}$ will be $= p\dot{x} \times PR^2 = p\dot{x} \times \overline{HP - c}|^2 = p\dot{x} \times \overline{HP^2 - c \times \overline{2HP - c}} = p\dot{x} \times \overline{HP^2 - c \times \overline{2RP + c}} = p\dot{x} \times \overline{r^2 - c^2 - x^2 - 2c \times RP}$; and $S = px \times \overline{r^2 - c^2 - \frac{1}{3}x^2 - \frac{2c}{x} \times RPCE} = p \times \overline{AE^2 - \frac{1}{3}x^2 \times x - 2c \times RPCE}$ $=$ the fruſtum generated by *RPCE*.

COROLLARY I.

When x becomes $= AE$, the theorem above becomes $\overline{\frac{1}{3}AE^3 - c \times ACE} \times 2p = \frac{1}{2}$ the ſpindle, and $\overline{\frac{1}{3}AE^3 - c \times ACE} \times 4p =$ the whole ſpindle.

Co-

That is, if $l = AE\frac{1}{2} =$ the length,
$c = EF =$ the central diſtance,
$a =$ the area ACE,
and $p = 3{\cdot}14159$ &c.

Then $\overline{\frac{1}{3}l^3 - ac} \times 4p =$ the whole ſpindle $ACBDA$.

EXAMPLE.

If the length of a circular ſpindle be 40, and its greateſt diameter 30; what is its ſolidity.

By the laſt problem the radius is $20\frac{5}{6}$, the central diſtance $5\frac{5}{6}$, and the length of the whole arc $53{\cdot}62508$; hence, $53{\cdot}62508 \times 20\frac{5}{6} =$ the double of the ſector FAB, (lines being ſuppoſed to be drawn from the center to A and B) and $40 \times 5\frac{5}{6} =$ the double of the triangle FAB, and conſequently $53{\cdot}62508 \times 20\frac{5}{6} - 40 \times 5\frac{5}{6} = \overline{53{\cdot}62508 \times 25 - 40 \times 7} \times \frac{5}{6} = 883{\cdot}8558 =$ the double of the ſegment ACB, $\frac{1}{4}$ of which, or $220{\cdot}9639$ is the half of that ſegment.

Then $\overline{\frac{1}{3}l^3 - ac} \times 4p = \overline{\frac{1}{3} \times 20^3 - 220{\cdot}9639 \times 5\frac{5}{6}} \times 4 \times 3{\cdot}1416 = \overline{\frac{4}{3} \times 20^2 - 220{\cdot}9639 \times 1\frac{1}{6}} \times 20 \times 3{\cdot}1416 = 275{\cdot}542 \times 62{\cdot}832 = 17312{\cdot}858 =$ the ſolidity required.

PRO-

COROLLARY II.

If from the $\frac{1}{2}$ ſpindle be taken the fruſtum, there will remain $p \times AE^2 \times AR - \frac{1}{3} \times \overline{AE^3 - x^3} - 2c \times APR = p \times \overline{\frac{1}{3}AR^2 \times \overline{3AE - AR}} - 2c \times APR$ for the ſegment PAQ of the ſpindle.

COROLLARY III.

If E coincide with F, the ſpindle will become a ſphere, c will vaniſh, and the theorems above become $\frac{4}{3}p \times GF^3 = \frac{1}{6}p \times GI^3$ for the whole ſphere; $p \times FH \times \overline{GF^2 - \frac{1}{3}FH^2}$ for the ſpheric fruſtum; and $\frac{1}{3}p \times GH^2 \times \overline{3GF - GH}$ for the ſpheric ſegment; and are the ſame with the theorems in problems 24, 25, and 27.

PROBLEM XXXI.

To find the content of the middle fruſtum or zone of a circular ſpindle.

RULE.

From the ſquare of half the length of the ſpindle take $\frac{1}{3}$ of the ſquare of half the length of the middle zone, and multiply the remainder into the ſaid half length of the zone, from the product ſubtract the product of the generating circular area and central diſtance; then the remainder drawn into 2 times 3·14159 &c. will be the content of the middle zone.

That is,

putting $l = ER =$ half the length of the zone,
$L = EA =$ half the length of the ſpindle,
$c = FE$ the central diſtance,
and $a =$ the generating area $RPYW$.

Then $\overline{LL-\frac{1}{3}ll\times l-ac}\times 2p =$ the zone $PYZQ$.

EXAMPLE.

If a caſk, in form of the middle fruſtum of a circular ſpindle, have its head diameter 24, bung diameter 32, and length 40 inches; how many ale gallons will it hold.

Here $CE-PR = \frac{32-24}{2} = 4 = TC$, $\frac{PT^2}{TC} = \frac{20^2}{4} = \frac{400}{4} = 100 = TK$; hence $100+4=104=$ the diameter, and 52 = the radius of the generating circle.

$FC-CE=52-16=36=EF$ the central diſtance, and $\sqrt{AF^2-FE^2}=\sqrt{52^2-36^2}=4\sqrt{13^2-9^2}=4\sqrt{88}=8\sqrt{22}$ $=EA=$ half the length of the whole ſpindle.

$PR\times RW = 12\times 40 = 480 =$ the area $RPTYW$, and the ſegment *PCY*, found by the table of circular ſegments, is 107·5185.

The ſum of theſe two is 587·5185 = the generating area *RPCYW*.

Whence $\overline{\overline{LL-\frac{1}{3}ll}\times\overline{l-ac}\times 2p} =$
$\overline{\overline{64\times22-\frac{1}{3}\times400\times20}-36\times587{\cdot}5185}\times2\times3{\cdot}14159$ &c. = $4342{\cdot}7879\times6{\cdot}28318$ &c. $=27286{\cdot}5411256=$ the ſolidity in inches.

Then $27286.5411256\div282=96{\cdot}7608$ ale gallons.

PROBLEM XXXII.

To find the content of the ſegment of a circular ſpindle.

RULE.

From 3 times half the length of the ſpindle ſubtract the height of the ſegment, and multiply the remainder into the ſquare of the ſaid height, from $\frac{1}{3}$ of the product ſubtract the double of the product of the generating area and central diſtance; then the remainder drawn into 3·14159 &c. will produce the content.

That is, $\overline{\overline{3EA-AR}\times AR^2-2FE\times APR}\times p=$ the ſolidity of *PAQ*.

EXAMPLE.

If the length of the whole ſpindle be 12, and its greateſt diameter 9; what will be the content of a ſegment of it whoſe height is 1.

$\frac{AE^2}{EC}=\frac{36}{4\frac{1}{2}}=\frac{72}{9}=8=EK$, $KE+EC=8+4\frac{1}{2}=12\frac{1}{2}=$ the diameter, and $6\frac{1}{4}=$ the radius of the generating circle; $FC-CE=6\frac{1}{4}-4\frac{1}{2}=1\frac{3}{4}=EF$ the central diſtance; $\sqrt{FP^2-PT^2}=\sqrt{\overline{6\frac{1}{4}}|^2-5^2}=\sqrt{\overline{\frac{5}{4}}|^2\times\overline{5^2-4^2}}=3\frac{3}{4}=TF$, $CF-FT=6\frac{1}{4}-3\frac{3}{4}=2\frac{1}{2}=TC$.

By the table of circular ſegments the half ſegment *PCT* is 8·736234375, to which adding $RT = 5 \times 2 = 10$, makes 18·736234375 = the area *RC*.

And in like manner, the half ſegment *ACE* is 19·886763281. Hence $ACE - RC = 19{\cdot}886763281 - 18{\cdot}736234375 = 1{\cdot}150528906 =$ the generating area *APR*.

Wherefore $\overline{\overline{3EA-AR} \times \frac{1}{3}AR^2 - 2FE \times ACE} \times p = \overline{17 \times \frac{1}{3} - 3\frac{1}{2} \times 1{\cdot}150528906} \times 3{\cdot}14159$ &c. $= 1{\cdot}6398154 95 \times 3{\cdot}14159$ &c. $= 5{\cdot}1516323 =$ the content required.

SECTION II.

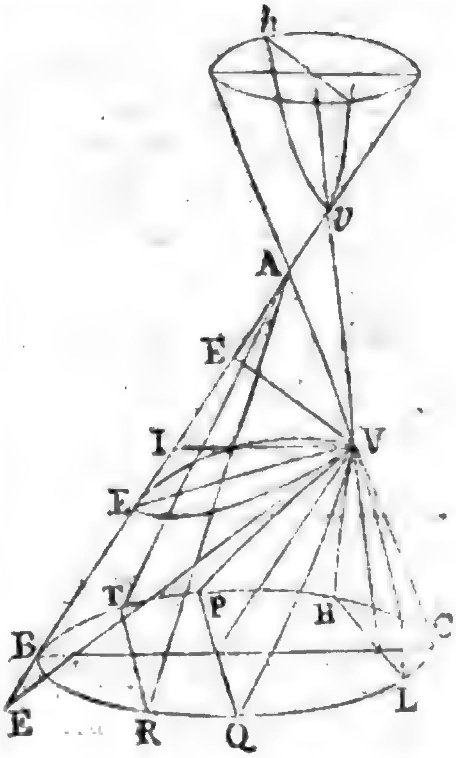
Of conic ſections, and of the figures ariſing from or depending on them.

GENERAL DEFINITIONS.

A Conic ſection is the plane ſurface made by a plane cutting a cone.

According to the different places or poſitions in which the cone is cut by the plane, there ariſe five different ſections, viz. an iſoſceles triangle, a circle, an ellipſe, a parabola, and an hyperbola; the three laſt of which only are peculiarly called conic ſections, becauſe the two firſt, the triangle and circle, are figures whoſe properties are known prior to thoſe of the cone.

If the cutting plane paſs through the vertex of the cone, the ſection will be an iſoſceles triangle *ATR*, whoſe ſides *RA*, *AT*, are each equal to the



ſlant height of the cone, and baſe *TR* is a chord of the circular baſe of the cone; and the cone will be cut into two pyramids *BTRA*, *CTRA*, each of the ſame height with the cone, and whoſe baſes are the ſegments *BTR*, *CTR*, of the circle *BTCRB*. If the plane paſs through the axe, the cone and its baſe will be divided equally in two.

If the plane cut the ſide of the cone below the vertex, the ſection will be one of the other four figures according to the various inclinations of the plane to the ſides of the cone.

If the cutting plane make no angle with the plane of the baſe, that is, if the plane be parallel to the baſe, the ſection *VI* will be a circle; for every ſection of a pyramid parallel to the baſe, is ſimilar to the baſe; and will cut off a fruſtum, *VCBI*.

If the plane, cutting the ſide, make an angle with the plane of the baſe, the ſection will be either an ellipſe, parabola, or hyperbola, and will cut off a hoof.

If the plane be inclined to the baſe in a leſs angle than the ſide is, the ſection *VE* will be an ellipſe, and will, therefore, cut the other ſide of the cone, produced if neceſſary, ſomewhere below the vertex *A*.

If the cutting plane be inclined to the baſe in the ſame angle as the ſide is, the ſection *VPQ* will be a parabola; which will not meet the oppoſite ſide *AB* of the cone, ſince it is parallel to it, by reaſon of its equal inclination to the baſe.

If the cutting plane make a greater angle with the plane of the baſe than the ſide of the cone doth, the the ſection *VLH* will be an hyperbola, and will, therefore, cut the oppoſite ſide of the cone, produced upwards, in *V* above the vertex. And if the ſide of the cone be continued in every direction through the vertex, ſo as to form above the vertex another cone equal to the original one; the ſection of the cutting plane, produced, and the upper cone will be an hyperbola equal to the lower one; and the two are called oppoſite hyperbolas.

The points *V*, *E*, *v*, where the cutting plane meets the upright triangular ſection *BCA*, are called the principal vertexes of the ſection.—Hence the ellipſe and hyperbola have each two vertexes, the parabola only one: or the parabola may alſo be ſaid to have two, by conſidering the one as infinitely diſtant from the other.

A line connecting the principal vertexes is the axe, axis, or tranſverſe diameter of the ſection.

The center of the ſection is the middle of the axe.—Hence the center of a parabola is infinitely diſtant from the vertex.

A diameter is any right line drawn through the center and terminated on each ſide by the curve, the interſections of the diameter and curve being the vertexes of the diameter.—Hence every diameter of the ellipſe and hyperbola have two vertexes; but of the parabola only one, unleſs we conſider the other as infinitely diſtant. And hence, alſo, all the diameters of a parabola are parallel to each other and infinite.

If a tangent to the curve be drawn through the vertex of any diameter, and another diameter be drawn parallel to the tangent, thoſe diameters are ſaid to be conjugates the one to the other.—The axes, or principal conjugate diameters, are perpendicular to each other.

An abſciſſa is any part of a diameter terminated at the vertex.

An ordinate to any diameter is a line contained between the diameter and the curve, and is parallel to the conjugate diameter, or to the tangent at the vertex.—The ordinates to the axe are perpendicular to it. And in the ellipſe and hyperbola every ordinate hath two abſciſſas, in the parabola only one.

The parameter of any diameter is a third proportional to that diameter and its conjugate.

The focus is the point of interſection of the axe and an ordinate, to it, which is equal to half the parameter of the axe.—The ellipſe and hyperbola have each two focuſes, the parabola only one.

A ſpindle is a body conceived to be generated by the revolution of the ſection about its double ordinate, and is denominated elliptic, parabolic, or hyperbolic, acccording as it is generated from the revolution of the ellipſe, parabola, or hyperbola.

A ſpheroid is the body conceived to be generated by the revolution of an ellipſe about its axe, and is denominated either prolate (oblong) or oblate according as the revolution is made about the tranſverſe axe or its conjugate. The axe about which the

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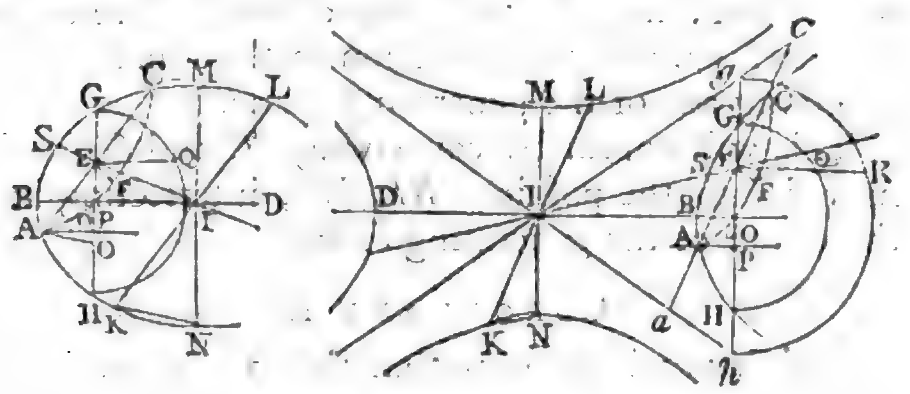
the revolution is made, is the fixed axe, the other is the revolving axe.

A conoid is a ſolid conceived to be generated from the revolution of the parabola or hyperbola about the tranſverſe axe, and is accordingly denominated either parabolic or hyperbolic. The former of theſe is, alſo, called a paraboloid; the latter, an hyperboloid.

PROPOSITION I.

If any ſolid formed by the revolution of a conic ſection about its axe, i. e. a ſpheroid, parabolid, or hyperboloid, be cut by a plane in any poſition; the ſection will be a conic ſection, and all the parallel ſections will be like and ſimilar figures.



DEMONSTRATION.

Let *ABC* be the generating ſection, or a ſection of the given ſolid through its axe *BD*, and perpendicular to the propoſed ſection *AFC*, their common interſection being *AC*; let *GH* be any other line meeting with the generating ſection in *G* and *H*, and cutting *AC* in *E*; and erect *EF* perpendicular to the plane *ABC* and meeting with the propoſed plane in *F*.

Then, if AC and GH be conceived to be moved continually parallel to themſelves, will the rectangle $AE \times EC$ be to the rectangle $GE \times EH$ always in a conſtant ratio; but if GH be perpendicular to BD, the points G, F, H will be in the circumference of a circle whoſe diameter is GH, ſo that $GE \times EH$ will be $= EF^2$; wherefore $AE \times EC$ will be to EF^2 always in a conſtant ratio; conſequently AFC is a conic ſection, and every ſection parallel to AFC will be of the ſame kind with and ſimilar to it. *Q. E. D.*

COROLLARY I.

The above conſtant ratio, in which $AE \times EC$ is to EF^2, is that of KI^2 to IN^2 the ſquares of the diameters of the generating ſection reſpectively parallel to AC, GH; that is, the ratio of the ſquare of the diameter parallel to the ſection to the ſquare of the revolving axe of the generating plane.

This will appear by conceiving AC and GH to be moved into the poſitions KL, MN, interſecting in I, the center of the generating ſection.

COROLLARY II.

And hence it appears that the axes AC and $2EF$ (ſuppoſing E now to be the middle of AC) of the ſection will be to each other as the diameter KL is to the diameter MN of the generating ſection.

COROLLARY III.

If the ſection of the ſolid be made ſo as to return into itſelf, it will evidently be an ellipſe. Which always happens in the ſpheroid, except when it is perpendicular to the axe, which poſition is alſo to be excepted in the other ſolids, the ſection being always then a circle: in the paraboloid the ſection is always an ellipſe excepting when it is parallel to the axe: and in the hyperboloid the ſection is always an ellipſe when its axe makes with the axe of the ſolid an angle greater than that made by the ſaid axe of the ſolid and the aſymptote of the generating hyperbola, the ſection being an hyperbola in all other caſes, but when thoſe angles are equal, and then it is a parabola.

COROLLARY IV.

But if the ſection be parallel to the fixed axe *BD*, it will be of the ſame kind with and ſimilar to the generating plane *ABC*; that is, the ſection parallel to the axe, in a ſpheroid, is an ellipſe ſimilar to the generating ellipſe; in the paraboloid, the ſection is a parabola ſimilar to the generating parabola; and in an hyperboloid, it is an hyperbola ſimilar to the generating hyperbola of the ſolid.

COROLLARY V.

In the ſpheroid, the ſection through the axe is the greateſt of the parallel ſections; but in the hyperboloid it is the leaſt; and in the paraboloid, all the ſections parallel to the axe are equal to one another. For the axe is the greateſt parallel chord line in the ellipſe, but the leaſt in the oppoſite hyperbolas, and all the diameters are equal in a parabola.

COROLLARY VI.

If the extremities of the diameters *KL*, *MN*, be joined by the line *KN*, and *AO* be drawn parallel to

KN and meeting with *GEH* in *O*, *E* being the middle of *AC*, or *AE* the ſemi-axe, and *GH* parallel to *MN*. Then *EO* will be equal to *EF* the other ſemi-axe of the ſection.

For, by ſimilar triangles, *KI* : *IN* :: *AE* : *EO*. Or upon *GH* as a diameter deſcribe a circle meeting *EQ* perpendicular to *GH* in *Q*; and it is evident that *EQ* will be equal to the ſemi-diameter *EF*.

COROLLARY VII.

Draw *AP* parallel to the axe *BD* of the ſolid and meeting the perpendicular *GH* in *P*. And it will be evident that, in the ſpheroid, the ſemi-axe $EF = EO$ will be greater than *EP*; but in the hyperboloid, the ſemi-axe $EF = EO$ of the elliptic ſection will be leſs than *EP*; and in the paraboloid, $EF = EO$ is always equal to *EP*.

SCHOLIUM.

The analogy of the ſections of an hyperboloid to thoſe of the cone are very remarkable, all the three conic ſections being formed by cutting an hyperboloid in the ſame poſitions as the cone is cut.

Thus let an hyperbola and its aſymptote be revolved together about the tranſverſe axe, the former deſcribing an hyperboloid and the latter a cone circumſcribing it; then let them be ſuppoſed to be both cut by a plane in any poſition; and the two ſections will be like, ſimilar, and concentric figures: that is, if the plane cut both the ſides of each, the ſections will be concentric, ſimilar ellipſes; if the cutting plane be parallel to the aſymptote or to the ſide of the cone, the ſections will be parabolas; and in all other poſitions, the ſections will be ſimilar and concentric hyperbolas.

That the ſections are like figures, appears from the foregoing corollaries; that they are concentric, will be evident when we conſider that Cc is $= Aa$, producing AC both ways to meet with the aſymptotes in a and c; and that they are ſimilar, or have their tranſverſe and conjugate axes proportional to each other, will appear thus: Produce GH both ways to meet the aſymptotes in g, h, and on the diameters GH, gh, deſcribe the ſemi-circles GQH, gRh, meeting EQR, drawn perpendicular to GH, in Q and R; EQ and ER being then evidently the ſemi conjugate axes, and EC, Ec, the ſemi-tranſverſe axes of the ſections: Now if GH and AC be conceived to be moved parallel to themſelves, $AE \times EC$ or CE^2 will be to $GE \times EH$ or EQ^2 in a conſtant ratio, or $\frac{CE}{EQ}$ will be a conſtant ratio; and ſince cE is as Eg, and aE as Eh, $aE \times Ec$ or cE^2 will be to $gE \times Eh$ or ER^2 in a conſtant ratio, or $\frac{cE}{ER}$ will be a conſtant ratio; but at an infinite diſtance from the vertex C and c coincide, or $EC = Ec$, as alſo $EG = Eg$, conſequently $EQ = ER$, and then $\frac{CE}{EQ}$ will be $= \frac{cE}{ER}$; but as theſe ratios are conſtant, if they be equal to each other in one place, they muſt be always ſo, and conſequently $CE : Ec :: QE : ER$.

And this analogy of the ſections will not ſeem ſtrange when we conſider that a cone is a ſpecies of the hyperboloid, or a triangle a ſpecies of the hyperbola, whoſe axes are infinitely little.

PROPOSITION II.

If SI be the ſemi-diameter belonging to the double ordinate AEC of the generating plane, AEC being the diameter of the ſection AFC conceived to be moved continually parallel to itſelf, and if x denote any part of the diameter SI intercepted by E the middle of AC and any given fixed point taken in SI; then will the ſection AFC be always as $A+Bx+Cxx$; A, B, C, being conſtant quantities; B in ſome caſes affirmative, and in others negative; C being affirmative in the hyperbola, negative in the ellipſe, and nothing in the parabola; and A may always be ſuppoſed to denote the diſtance of the given fixed point from the vertex S.

DEMONSTRATION.

In any conic ſection, AC^2 is as $A+Bx+Cxx$; but all the parallel ſections are like and ſimilar figures, and ſimilar plane figures are as the ſquares of their like dimenſions; therefore the ſection AFC is as AC^2, that is, as $A+Bx+Cxx$. *Q. E. D.*

COROLLARY.

If the given fixed point, where x begins, coincide with the vertex S, then will A be equal to nothing, and the ſection will be as $Bx \pm Cxx$ or as $x \pm Dxx$ in the hyperbola and ellipſe, and as Bx or as x in the parabola.

SECT.

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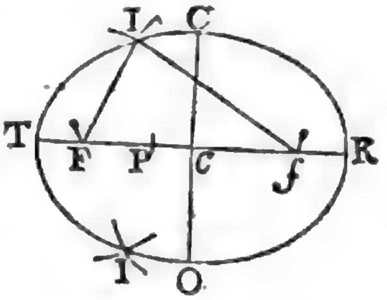
SECTION III.

Of the ellipſe, and the figures generated from it.

PROBLEM I.

To deſcribe an ellipſe; having the tranſverſe and conjugate axes given.

RULE.*



1. DRAW the tranſverſe *TR* and its conjugate *CO* biſecting each other perpendicularly in the center *c*.

2. With the radius *Tc* and center *C*, deſcribe an arc cutting *TR* in the points *F*, *f*, which will be the focuſes of the ellipſe.

3. Take any point *P* in the tranſverſe, then with the radiuſes *TP*, *PR*, and centers *F*, *f*, deſcribe two arcs interſecting in *I*; ſo will the point *I* be in the curve or circumference of the ellipſe.

And thus, by aſſuming ſeveral points *P* in the tranſverſe, there will be found as many points in the curve as we ſhall chuſe; through all which let the curve line be drawn.

The ſame performed from the ſame principle by means of a ſtring.

Having found the focuſes *F*, *f*, take a thread of the length of the tranſverſe; faſten its ends with pins in the focuſes; then ſtretch the thread *FIf* and it will reach to *I* in the curve: and by moving a pencil round within the thread, keeping it always ſtretched, it will trace out the curve.

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* The truth of this conſtruction will appear by obſerving that the tranſverſe axe is equal to the ſum of two lines drawn from the focuſes to meet in any point in the curve.

PROBLEM II.

In an ellipſe, to find any two conjugate diameters, an ordinate to one of them, and its abſciſſa, one from another; viz. having any three of them given, to find the fourth.

CASE I. * *To find the ordinate.*

RULE.

As any diameter:
Is to its conjugate::
So is the mean proportional between the two abſciſſas or ſegments of the diameter:
To the ordinate.

That is, as $d : c :: \sqrt{x \times \overline{d-x}} : \frac{c}{d}\sqrt{x \times \overline{d-x}} = y$; putting d for the diameter, c its conjugate, and x an abſciſſa to the ordinate y.

EXAMPLE.

If the tranſverſe be 35, the conjugate 25, and the abſciſſa 7; what is the ordinate.

Here $\frac{c}{d}\sqrt{x \times \overline{d-x}} = \frac{25}{35}\sqrt{7 \times \overline{35-7}} = \frac{5}{7}\sqrt{7 \times 28} = \frac{5}{7}\sqrt{7^2 \times 4}$ $= \frac{5 \times 7 \times 2}{7} = 10 =$ the ordinate.

CASE

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* The values of the ſeveral quantities in the four caſes of this problem are found from the general equation expreſſing the property of the curve, viz. $dd : cc :: x \times \overline{d-x} : yy$.

CASE II. *To find the abſciſſas.*

RULE.

As the conjugate:
Is to the diameter::
So is the ſquare root of the difference of the ſquares of the ordinate and ſemi-conjugate:
To the diſtance between the ordinate and center.

Then that diſtance being added to and ſubtracted from the ſemi-diameter, will give the two abſciſſas.

That is, $\frac{1}{2}d \pm \frac{d}{c}\sqrt{\frac{1}{4}cc-yy} = x$.

EXAMPLE.

What are the two abſciſſas of the ordinate 10, the diameters being 35 and 25.

Here $\frac{1}{2}d \pm \frac{d}{c}\sqrt{\frac{1}{4}cc-yy} = \frac{35}{2} \pm \frac{35}{25}\sqrt{\overline{\frac{25}{2}}|^2 - 10^2} = \frac{35}{2} \pm \frac{7}{2}\sqrt{5^2-4^2} = \frac{35 \pm 21}{2} = 28$ and 7 the two abſciſſas required.

CASE III. *To find the conjugate.*

RULE.

As the mean proportional between the two abſciſſas:
Is to the ordinate::
So is the diameter:
To its conjugate.

That is, as $\sqrt{x \times \overline{d-x}} : y :: d : \frac{dv}{\sqrt{dx-xx}} = c$.

EXAMPLE.

What is the conjugate to the diameter 35; the abſciſſa to an ordinate of 10 being 7.

Here $\frac{dy}{\sqrt{\overline{d-x}\times x}} = \frac{35\times 10}{\sqrt{28\times 7}} = \frac{350}{14} = 25$ the conjugate required.

CASE IV. *To find the diameter.*

RULE.

To or from the ſemi-conjugate, according as the leſs or greater abſciſſa is uſed, add or ſubtract the root of the difference of the ſquares of the ordinate and ſemi-conjugate: Then, as the ſquare of the ordinate is to the product of the conjugate and abſciſſa, ſo is the ſum or difference above found to the diameter.

That is, $\frac{cx}{yy}\times\overline{\tfrac{1}{2}c\mp\sqrt{\tfrac{1}{4}cc-yy}}=d$.

EXAMPLE.

If an ordinate and its leſs abſciſſa be 10 and 7; what is the diameter, ſuppoſing the conjugate to be 25.

Here $d=\frac{cx}{yy}\times\overline{\tfrac{1}{2}c+\sqrt{\tfrac{1}{4}cc-yy}}=\frac{25\times 7}{10\times 10}\times\overline{\frac{25}{2}+\sqrt{\overline{\tfrac{25}{2}}\Big|^{2}-10^{2}}}=$ $\frac{7}{4}\times\frac{25+15}{2}=7\times 5=35$ the tranſverſe.

PROBLEM III.

To find the length of the circumference of an ellipſe.

RULE.

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*RULE I.

Multiply the circumference of the circumſcribing circle into the ſeries $1-\frac{d}{2.2}-\frac{3dd}{2.2.4.4}-\frac{3.3.5d^3}{2.2.4.4.6.6}-\frac{3.3.5.5.7d^4}{2.2.4.4.6.6.8.8}$ &c. where $d=$ the difference between an unit and the ſquare of the quotient of the leſs axe divided by the greater, and the product will be the periphery of the ellipſe ſought.

Ex-

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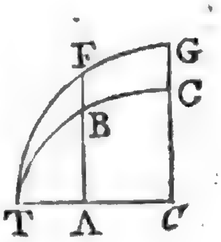
*DEMONSTRATION.



Let the ſemi-tranſverſe axe Tc be called a; the ſemi-conjugate cC, c; the ordinate AB, y; and its diſtance cA from the center, x; alſo the arc CB, z.

Then ſince, by the nature of the curve, y is $=\frac{c}{a}\sqrt{aa-xx}$, $\dot{y}$ will be $=\frac{-c\dot{x}x}{a\sqrt{aa-xx}}$; and conſequently $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}=\frac{\dot{x}\sqrt{a^2-\frac{a^2-c^2}{aa}\times x^2}}{\sqrt{aa-xx}}=\frac{\dot{x}\sqrt{aa-dxx}}{\sqrt{aa-xx}}$ (by writing d for $\frac{aa-cc}{aa}$ or $1-\frac{cc}{aa}$) $=\frac{a\dot{x}\sqrt{1-\frac{dxx}{aa}}}{\sqrt{aa-xx}}=\frac{a\dot{x}}{\sqrt{aa-xx}}\times:1-\frac{dx^2}{2a^2}-\frac{d^2x^4}{2.4a^4}-\frac{3d^3x^6}{2.4.6a^6}$ &c. by throwing the numerator into a ſeries. But the fluent of $\frac{a\dot{x}}{\sqrt{aa-xx}}$ is $=$ the correſponding circular arc GF deſcribed with the center c and radius cT, which call A: then, *per* page 66 *Coteſii Harmonia Menſurarum*, the fluent comes out $A-\frac{d}{2a^2}B-\frac{d^2}{2.4a^4}C-\frac{3d^3}{2.4.6a^6}D$ &c. wherein $B=\frac{aaA-x\sqrt{aa-xx}}{2}$, $C=\frac{3aaB-x^3\sqrt{aa-xx}}{4}$, $D=\frac{5aaC-x^5\sqrt{aa-xx}}{6}$, &c. for the length of the arc CB.

But

EXAMPLE.

Required the periphery of an ellipſe whoſe axes are 24 and 18.

Here $1-\overline{\frac{18}{24}}\Big|^2 = 1-\overline{\frac{3}{4}}\Big|^2 = \frac{7}{16} = \cdot 4375 = d$.

Then

But when A arrives at T, x is $= a$, and $\sqrt{aa-xx} = 0$; hence the values of the quantities B, C, D, &c. become barely $B = \frac{aa}{2}A$, $C = \frac{3aa}{4}B = \frac{3a^4}{2.4}A$, $D = \frac{5aa}{6}C = \frac{3.5a^6}{2.4.6}A$, &c. And, conſequently, the ſeries above becomes $A\times : 1 - \frac{d}{2.2} - \frac{3d^2}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6}$ &c. for $\frac{1}{4}$ of the elliptic periphery, where A is $\frac{1}{4}$ of the periphery of the circle, or $=$ the whole circumference of the ellipſe when A is that of the circle. *Q. E. D.*

COROLLARY I.

Hence the peripheries of ſimilar ellipſes are to each other as thoſe of their circumſcribed circles, or as any two ſimilar diameters of thoſe ellipſes.

COROLLARY II.

If a be the leſs ſemi-axe, and $c =$ the greater, the fluxion of the arc will be $\dot{z} = \dot{x}\sqrt{\frac{aa+dxx}{aa-xx}}$; and if $a = 1$, and $c = \sqrt{2}$, the ſame fluxion $\dot{z}$ will be $= \dot{x}\sqrt{\frac{1+xx}{1-xx}} = \dot{x}\frac{1+xx}{\sqrt{1-x^4}} = -\frac{\dot{x}}{\sqrt{1-x^4}} + \frac{x^2\dot{x}}{\sqrt{1-x^4}}$.—And if we write v for xx, we ſhall have $\dot{z} = \frac{\dot{v}}{2\sqrt{v}}\sqrt{\frac{1+v}{1-v}}$, as Mr *Landen* found in page 142 of his *Math. Lucub.*—Or, by writing w^n for x, we ſhall have univerſally $\dot{z} = nw^{n-1}\dot{w}\sqrt{\frac{1+w^{2n}}{1-w^{2n}}}$. Which theorems are of uſe in determining ſome fluents.

Then the 2d term $A=\frac{d}{4}$ = ·10938

the 3d — $B=\frac{3d}{4.4}A$=·00897

the 4th — $C=\frac{3.5d}{6.6}B$=·00164

the 5th — $D=\frac{5.7d}{8.8}C$=·00039

the 6th — $E=\frac{7.9d}{10.10}D$=·00011

the 7th — $F=\frac{9.11d}{12.12}E$=·00003

the 8th — $G=\frac{11.13d}{14.14}F$=·00001

their ſum is ·12053, which taken from the firſt term 1, leaves ·87947 for the value of the ſeries; and being drawn into 3·1416×24 the periphery of the circumſcribing circle, gives 66·31056 for that of the ellipſe required.

*RULE II.

Multiply ½ the ſum of the two axes by 3·1416, and the product will be the circumference nearly.

That is, $\frac{t+c}{2}\times p$ = the circumference nearly, putting t for the tranſverſe, c for its conjugate axe, and p for 3·1416.

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[*] It will be evident that this rule will not differ much from the truth, if it be conſidered that this arithmetical mean between the axes exceeds their geometrical mean, and that the geometrical mean (as will be proved hereafter) is the diameter of a circle equal in area to the ellipſe, which circle is of a leſs ambit than the ellipſe, or any other figure, of the ſame area.

EXAMPLE.

Let there be here taken the ſame example as before, in which the axes are 24 and 18.

Then $\frac{24+18}{2}\times 3\cdot1416=21\times3\cdot1416=65\cdot9736=$ the circumference nearly.

*RULE III.

Multiply the ſquare root of ½ the ſum of the ſquares of the two axes by 3·1416, and the product will be the circumference nearly.

That is, $p\sqrt{\frac{tt+cc}{2}}=$ the circumference nearly.

EXAMPLE.

Taking again the ſame example.

We have $\sqrt{\frac{24^2+18^2}{2}}\times3\cdot1416=15\sqrt{2}\times3\cdot1416=$ $21\cdot2132\times3\cdot1416=66\cdot6433=$ the circumference nearly.

RULE

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[*] Call the infinite ſeries in rule 1, S; viz. $S=1-\frac{d}{2.2}-\frac{3d^2}{2.2.4.4}-\frac{3.3.5d^3}{2.2.4.4.6.6}$ &c. then ſince $\sqrt{1-\frac{1}{2}d}$ is $=1-\frac{d}{2.2}-\frac{d^2}{2^3.4}-\frac{3d^3}{2^4.4.6}$ &c. we ſhall have $S-\sqrt{1-\frac{1}{2}d}=-\frac{d^2}{64}-\frac{3d^3}{256}$ &c. Wherefore rejecting this laſt remaining ſeries, on account of its ſmallneſs, we have $S=\sqrt{1-\frac{1}{2}d}$ nearly, and conſequently the circumference of the ellipſe $C=c\sqrt{1-\frac{1}{2}d}$ nearly, c being the circumference of the circumſcribed circle. Or $C=pt\sqrt{1-\frac{1}{2}d}=pt\sqrt{1-\frac{1}{2}+\frac{cc}{2tt}}=pt\sqrt{\frac{1}{2}+\frac{cc}{2tt}}=p\sqrt{\frac{tt+cc}{2}}$, which is the 3d rule, p being $=3\cdot14159$ &c.

Again

RULE IV.*

Find the ſquare root of $\frac{1}{2}$ the ſum of the ſquares of the axes as in the laſt rule, and call it A.

To 3 times the greater axe add its parameter, divide the ſum by 4, and call the quotient B.

Then multiply the difference between B and 3 times A by $1{\cdot}5708\left(=\frac{3{\cdot}1416}{2}\right)$, and the product will be the periphery nearer than by the laſt rule.

That is, $\frac{1}{2}p\times\overline{3\sqrt{\frac{tt+cc}{2}}-\frac{3t+P}{4}}=$ the periphery, P being the parameter.

EXAMPLE.

Taking the ſame example as in the former rules,

We have $A=\sqrt{\frac{24^2+18^2}{2}}=15\sqrt{2}=21{\cdot}2132$.

As $24:18::18:13\frac{1}{2}=$ the parameter, by the definition.

And $\frac{3\times24+13\frac{1}{2}}{4}=21\frac{1}{8}=21{\cdot}375=B$.

Then $\overline{3A-B}\times1{\cdot}5708=\overline{21{\cdot}2132\times3-21{\cdot}375}\times1{\cdot}5708$ $=42{\cdot}2646\times1{\cdot}5708=66{\cdot}3892=$ the circumference nearly.

RULE

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* Again, $S-1+\frac{1}{4}d=-\frac{3d^2}{64}-\frac{5d^3}{256}$ &c. but $3S-3\sqrt{1-\frac{1}{2}d}=-\frac{3d^2}{64}-\frac{9d^3}{256}$ &c. wherefore $2S-3\sqrt{1-\frac{1}{2}d}+1-\frac{1}{4}d=-\frac{d^3}{64}$ &c. hence $S=\frac{\overline{3\sqrt{1-\frac{1}{2}d}-1+\frac{1}{4}d}}{2}$ nearly, and conſequently $C=\frac{1}{2}pt\times\overline{3\sqrt{1-\frac{1}{2}d}-\frac{4-d}{4}}$ $=\frac{1}{2}p\times\overline{3\sqrt{\frac{tt+cc}{2}}-\frac{3tt+cc}{4t}}=\frac{1}{2}p\times\overline{3\sqrt{\frac{tt+cc}{2}}-\frac{3t+P}{4}}$ nearly, where P is the parameter of the axe. And this is the 4th rule.

Farther,

RULE V.[*]

Find *A* and *B* as in the laſt rule.

Divide the difference of the ſquares of the axes by the ſquare of the greater axe, multiply the ſquare of the quotient by $\frac{1}{16}$ of the greater axe, and call the reſult *C*.

Then from 5 times *A* take 3 times *B*, to the remainder add *C*, multiply the ſum by 1·5708, and the product will give the circumference of the ellipſe ſtill nearer.

Or,

From 5 times *A* ſubtract $\frac{1}{16}$ of the ſum of 35 times the greater axe and 14 times its parameter, to the remainder add $\frac{1}{16}$ of the product ariſing from the multiplication of the ſaid parameter by the quotient of the ſquare of the leſs axe divided by that of the greater, and multiply the ſum by 1·5708.

That is, $\frac{1}{2}p\times\overline{5\sqrt{\frac{tt+cc}{2}}-3\times\frac{3t+p}{4}+\overline{\frac{tt-cc}{tt}}\Big|^{2}\times\frac{t}{16}}$, or $\frac{1}{2}p\times\overline{5\sqrt{\frac{tt+cc}{2}}-\frac{35t+14p}{16}+\frac{pcc}{16tt}}=$ the periphery very nearly.

Ex-

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[*] Farther, $S-1+\frac{d}{4}+\frac{3d^2}{64}=-\frac{5d^3}{256}$ &c. or $\frac{4}{5}S-\frac{4}{5}+\frac{d}{5}+\frac{3d^2}{80}=-\frac{d^3}{64}$, which taken from the laſt approximation, and reduced, gives $S=\frac{1}{2}\times\overline{5\sqrt{1-\frac{1}{2}d}-3\times\frac{4-d}{4}+\frac{dd}{16}}$; and conſequently $C=\frac{1}{2}pt\times\overline{5\sqrt{1-\frac{1}{2}d}-3\times\frac{4-d}{4}+\frac{dd}{16}}=\frac{1}{2}p\times\overline{5\sqrt{\frac{tt+cc}{2}}-3\times\frac{3t+P}{4}+\overline{\frac{tt-cc}{tt}}\Big|^{2}\times\frac{t}{16}}$ $=\frac{1}{2}p\times\overline{5\sqrt{\frac{tt+cc}{2}}-\frac{35t+14P}{16}+\frac{Pcc}{16tt}}$. Which is the 5th rule. And thus we may proceed to any degree of accuracy required.

EXAMPLE.

Taking again the ſame example as before, we have

$A = 21{\cdot}2132$,

$B = 21{\cdot}375$,

$C = \overline{\frac{24^2-18^2}{24}}\Big|^2 \times \frac{24}{16} = \overline{\frac{4^2-3^2}{4^2}}\Big|^2 \times \frac{3}{2} = \frac{7^2\times3}{16^2\times2} = {\cdot}2871.$

Then $\overline{5A-3B+C} \times 1{\cdot}5708 = \overline{106{\cdot}066-64{\cdot}125+{\cdot}2871} \times 1{\cdot}5708 = 42{\cdot}2281 \times 1{\cdot}5708 = 66{\cdot}3319 =$ the circumference very near.

Or, to uſe the latter part of the rule,

$\overline{5A - \frac{35\times24-14\times13\frac{1}{2}}{16} + \frac{13\frac{1}{2}\times18^2}{16\times24^2}} \times 1{\cdot}5708 = 42{\cdot}2281 \times 1{\cdot}5078 = 66{\cdot}3319$ the ſame as before.

RULE VI.

Take $\frac{1}{2}$ the ſum of the quantities in the ſecond and third rules, and it will give the circumference very nearly.*

That is, $\frac{1}{2}p \times \overline{\frac{t+c}{2} + \sqrt{\frac{tt+cc}{2}}} =$ the periphery.

EXAMPLE.

Taking, ſtill, the ſame example,

The number found by the 2d rule is 65·9736,

And by the 3d rule is 66·6433,

their ſum is 132·6169, the half of which is 66·3084 for the circumference, and is nearer to the true number found by rule 1 than any of the other approximations.

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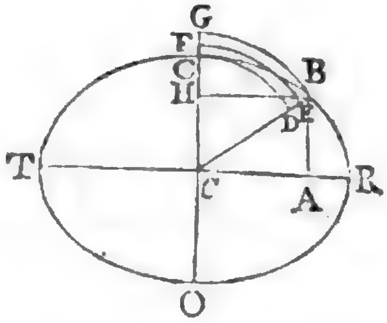
* For the one being nearly as much too great as the other is too little, $\frac{1}{2}$ their ſum muſt be near the truth.

PROBLEM IV.

To find the length of any arc of an ellipſe.

* RULE I.



If a denote the ſemi-tranſverſe Tc, c the ſemi-conjugate cC, and x the diſtance cA of the ordinate AB from the center, then will the arc CB be $= x \times : 1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{40a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}}x^6$ &c.

And the ſeries will be the ſame if x be conſidered as an ordinate to either axe $2c$, the other axe being $2a$, the arc beginning at the vertex of the axe $2c$ and terminated by the ordinate x.

Ex-

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* DEMONSTRATION.

By proceeding as in the inveſtigation of problem 3, we have the fluxion of the curve $\dot{z} = \dot{x}\sqrt{\frac{aa - dxx}{aa - xx}} = \dot{x} \times : 1 + \frac{c^2}{2a^4}x^2 + \frac{4a^2c^2 - c^4}{8a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{16a^{12}}x^6$ &c. reſtoring c and exterminating d; and the fluent is $z = x \times : 1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{40a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}}x^6$ &c.

COROLLARY I.

When x becomes $= a$, the rule becomes $a \times : 1 + \frac{c^2}{6a^2} + \frac{4a^2c^2 - c^4}{40a^4} + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^6}$ &c. for one-fourth of the periphery of the ellipſe. *Q. E. D.*

COROLLARY II.

By ſeppoſing $a = c$, the ellipſe will become a circle, and the general ſeries for the elliptic arc will become for the circular arc $x \times : 1 + \frac{x^2}{6a^2} + \frac{3x^4}{40a^4} + \frac{5x^6}{112a^6}$ &c. $= x \times : 1 + \frac{x^2}{2.3a^2} + \frac{3x^4}{2.4.5a^4} + \frac{3.5x^6}{2.4.6.7a^6}$ &c. the ſame as was determined at page 88.

EXAMPLE.

If the two axes be 24 and 18, and the diſtance *cA* be 3; Required the length of the arc *BC*.

Here $a=12$, $c=9$, and $x=3$.

Then the 1ſt term = — — 1·00000

2d term $\frac{c^2}{6a^4}x^2=A=$ — — 0·00586

3d term $B=3\times\frac{4a^2-c^2}{20a^4}x^2A=$ — 0·00019

4th term $C=\frac{5}{14a^4}\times\frac{8a^4-4a^2c^2+c^4}{4a^2-c^2}x^2B=$ — 0·00001

the ſum is 1·00606

And $3\times1{\cdot}00606=3{\cdot}01818=$ the arc required.

[*] RULE II.

Find the length of the circular arc *FE* intercepted by *Cc*, *cB*, and whoſe radius is ½ the ſum of the lines *Cc*, *cB*; and it will be equal to the elliptic arc *BC* nearly.

EXAMPLE.

Taking here the ſame example,

We have $RA=9$, and $AT=15$.

Hence $Tc^2:cC^2::TA\times AR:AB^2=\frac{9^2\times9\times15}{12\times12}=\frac{9\times9\times15}{16}$, and

$BC=\sqrt{cA^2+AB^2}=\sqrt{9+\frac{9\times9\times15}{16}}=\frac{3}{4}\sqrt{151}=9{\cdot}21616$.

---

[*] For the circular arc *FE* is equal to the arithmetical mean between the circular arcs *BG*, *CD*, deſcribed with the radiuſes *Bc*, *cC*; to which it is evident the elliptic arc muſt be nearly equal, and ſo much the nearer, the leſs the arc itſelf is.—Rule 2 of the laſt problem is a particular caſe of this general rule.

This rule will alſo be the more exact the nearer the axes of the ellipſe approach towards an equality.

Then $\frac{Cc+cB}{2}=\frac{18{\cdot}21616}{2}=9{\cdot}10808=$ the radius of the circular arc *EF*.

But $\frac{HB}{Bc}=\frac{3}{9{\cdot}21616}={\cdot}325515=$ the ſine of the angle *CcB* or arc *EF* to the radius 1, to which belong 18·9968 degrees.

Wherefore, by rule 1 prob. 6 ſect. 1 part 2, $\cdot01745\times18{\cdot}9968\times9{\cdot}10808=3{\cdot}0192=$ the circular arc *EF*, or elliptic arc *BC* nearly.

*RULE III.

Divide the difference of the ſquares of the ſemi-axes by the ſquare of the greater, and call the quotient *q*.

Then divide the difference between the ſquare of the greater ſemi-axe and the product of $\frac{1}{3}$ of the ſquare of the diſtance *cA* of the ordinate from the center multiplied into q, by the difference between the ſquare of the ſaid greater ſemi-axe and $\frac{1}{5}$ of the ſquare of the ſaid diſtance; and the root of the quotient multiplied into the ſaid diſtance will be the arc nearly.

---

* DEMONSTRATION.

By the firſt rule the arc *A* is equal to $x\times:1+\frac{c^2}{6a^4}x^2+\frac{4a^2c^2-c^4}{40a^8}x^4+\frac{8a^4c^2-4a^2c^4+c^6}{112a^{12}}x^6$ &c. but in the inveſtigation of that rule it appears that $\sqrt{\frac{aa-dxx}{aa-xx}}$ is $=1+\frac{c^2}{2a^4}x^2+\frac{4a^2c^2-c^4}{8a^8}x^4+\frac{8a^4c^2-4a^2c^4+c^6}{16a^{12}}$ &c. hence it is evident that $x\sqrt{\frac{aa-\frac{1}{3}dxx}{aa-\frac{1}{3}xx}}\times:1+\frac{c^2}{6a^4}x^2+\frac{4a^2c^2-c^4}{72a^8}x^4+\frac{8a^4c^2-4a^2c^4+c^6}{432a^{12}}x^6$ &c. and conſequently $A=x\sqrt{\frac{aa-\frac{1}{3}dxx}{aa-\frac{1}{3}xx}}$ nearly. *Q. E. D.*

COROLLARY.

When the two axes are equal to each other, the ellipſe will become a circle, *d* will be $=0$, and the rule will become $x\sqrt{\frac{aa}{aa-\frac{1}{3}xx}}$ for the circular arc.

That is, $x\sqrt{\frac{aa-\frac{1}{5}qxx}{aa-\frac{1}{5}xx}}=$ the arc, where $a=Tc$ the greater ſemi-axe, $x=cA$, and $q=\frac{aa-cc}{aa}=1-\frac{cc}{aa}$, c being the leſs ſemi-axe cC.

EXAMPLE.

Taking again the ſame example,

We have $1-\frac{cc}{aa}=1-\overline{\frac{9}{12}}\Big|^2=1-\overline{\frac{3}{4}}\Big|^2=\frac{7}{16}=q$.

And $x\sqrt{\frac{aa-\frac{1}{5}qxx}{aa-\frac{1}{5}xx}}=3\sqrt{\frac{12^2-\frac{1}{5}\times\frac{7}{16}\times 3^2}{12^2-\frac{1}{5}\times 3^2}}=3{\cdot}017898=$ the arc required nearly.

* RULE IV.

Call the quantity found by the laſt rule B.

---

* DEMONSTRATION.

For ſince $A=xx:1+\frac{c^2}{6a^4}x^2+\frac{4a^2c^2-c^4}{40a^8}x^4+\frac{8a^4c^2-4a^2c^4+c^6}{112a^{12}}x^6$ &c. and $B=xx:1+\frac{c^2}{6a^4}x^2+\frac{4a^2c^2-c^4}{72a^8}x^4+\frac{8a^4c^2-4a^2c^4+c^6}{432a^{12}}x^6$ &c. we have $A-B=xx:\frac{4a^2c^2-c^4}{90a^8}x^4+\frac{40a^4c^2-20a^2c^4+5c^6}{756a^{12}}$ &c. but $\frac{4}{5}\times\overline{B-xx\times 1+\frac{c^2}{6a^4}x^2}=xx:\frac{4a^2c^2-c^4}{90a^8}x^4+\frac{8a^4c^2-4a^2c^4+c^6}{540a^{12}}x^6$ &c. therefore $A-B-\frac{4}{5}\times\overline{B-xx\times 1+\frac{c^2x^2}{6a^4}}=xx:\frac{8a^4c^2-4a^2c^4+c^6}{210a^{12}}x^6$ &c. and, rejecting the ſeries, $A=\frac{1}{5}\times\overline{9B-4xx\times 1+\frac{c^2x^2}{6a^4}}=\frac{1}{5}x\times\overline{9\sqrt{\frac{aa-\frac{1}{5}dxx}{aa-\frac{1}{5}xx}}-4\times 1+\frac{c^2x^2}{6a^4}}$ nearly. Q. E. D.

Multiply the diſtance *cA* into the leſs ſemi-axe, divide the product by the ſquare of the greater, to $\frac{1}{6}$ of the ſquare of the quotient add 1, multiply the ſum into the diſtance *cA*, and call the product *C*.

Then from 9 times *B* take 4 times *C*, divide the remainder by 5, and the quotient will be the arc very near.

That is, $A = \frac{9B-4C}{5} = \frac{1}{5} \times 9x \sqrt{\frac{aq-\frac{1}{5}qxx}{aa-\frac{1}{5}xx}} - 4x \times \overline{1+\frac{c^2x^2}{6a^4}}$.

EXAMPLE.

Taking ſtill the ſame example,

The Quantity found by the laſt rule is 3·017898 $= B$.

And $x \times \overline{1+\frac{c^2x^2}{6a^4}} = \times \overline{1+\frac{9^2 \times 3^2}{6 \times 12^4}} = 3 \times \overline{1+\frac{3}{2 \times 4^4}} = 3{\cdot}0175778$ $= C$.

Then $\frac{9B-4C}{5} = \frac{27{\cdot}161082-12{\cdot}070212}{5} = \frac{15{\cdot}09077}{5} =$ $3{\cdot}018154 =$ the arc very nearly.

RULE V.

As the ſum of 15 times the parameter of the axe, at whoſe vertex the arc begins, and a third proportional to the ſaid axe, the abſciſſa, and the difference between 9 times the ſaid axe and 21 times its parameter; is to the ſum of 15 times the parameter and a third proportional to the axe, the abſciſſa, and the dif-

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COROLLARY.

When a is $= c$, then the rule becomes $\frac{1}{5}x \times 9 \sqrt{\frac{aa}{aa-\frac{1}{5}xx}} - 4 \times \frac{6aa+xx}{6aa}$ for the circular arc difference between 19 times the axe and 21 times its parameter; ſo is the ordinate, to the length of the arc, very nearly.*

That is, $\dfrac{15p+\dfrac{19C-21p}{C}\times y}{15p+\dfrac{9C-21p}{C}\times y}\times x=$ the arc CB, where $C=$ the whole axe CO at whoſe vertex the arc commences,

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*DEMONSTRATION.

Since y is $=c-\dfrac{c\sqrt{aa-xx}}{a}=\dfrac{cx^2}{2a^2}+\dfrac{cx^4}{8a^4}$ &c. it is evident that a fraction of this form $x\times\dfrac{A+\overline{B+1}.y}{A+By}$, when expanded in a ſeries, will produce a ſeries of the ſame form with that expreſſing the length of the arc in the firſt rule, and which being put equal to it will afford equations for determining the values of A and B.

Thus $x\times\dfrac{A+\overline{B+1}.y}{A+By}$ is $=x\times\dfrac{A+\overline{B+1}\times:\dfrac{cx^2}{2a^2}+\dfrac{cx^4}{8a^4}\text{ \&c.}}{A+B\times:\dfrac{cx^2}{2a^2}+\dfrac{cx^4}{8a^4}\text{ \&c.}}=x\times:1+\dfrac{cx^2}{2Aa^2}+\dfrac{Ac-2Bc^2}{8A^2a^4}x^4$ &c. which let be put $=x\times:1+\dfrac{c^2x^2}{6a^4}+\dfrac{4a^2c^2-c^4}{8a^8}x^4$ &c. Then, by equating the correſponding terms, we obtain $\dfrac{c}{2Aa^2}=\dfrac{c^2}{6a^4}$, or $A=\dfrac{3a^2}{c}=\frac{3}{2}p$; and $\dfrac{Ac-2Bc^2}{8A^2a^4}=\dfrac{4a^2c^2-c^4}{8a^8}$, hence $B=\dfrac{9c^2-21a^2}{10cc}=\dfrac{9c-\dfrac{21aa}{c}}{10c}=\dfrac{9C-21p}{10C}$, putting $C=2c$, and $p=$ its parameter $\dfrac{2aa}{c}$.

Conſequently

$$x\times\frac{A+\overline{B+1}.y}{A+By}\text{ is }=x\times\frac{\frac{3}{2}p+\dfrac{19C-21p}{10C}y}{\frac{3}{2}p+\dfrac{9C-21p}{10C}y}=x\times\frac{15p+\dfrac{19C-21p}{C}y}{15p+\dfrac{9C-21p}{C}y}.\ Q.E.D.$$

mencees, $p=$ its parameter $=\frac{TR^2}{CO}$, $y=$ the abſciſſa CH, and $x=$ the ordinate HB.

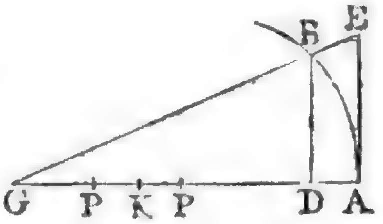
EXAMPLE.

Taking, ſtill, the ſame example, in which $c=9$, $a=12$, $C=18$, $p=\frac{12\times24}{9}=32$, and $x=3$; hence $y=c-\frac{c\sqrt{aa-xx}}{a}=9-\frac{9\sqrt{12^2-3^2}}{12}=9-\frac{9\sqrt{15}}{4}=9\times\frac{4-\sqrt{15}}{4}=$ $\cdot2857874715$.

Then

COROLLARY I.

Hence, in the ellipſe AB whoſe vertex is A, either axe AK, and latus rectum AP; if there be drawn the ordinate BD, and PG be taken $=\frac{1}{2}AP+\frac{19AK-21AP}{10AK}\times AD$, and there be drawn the right line GBE meeting the tangent AE in E; then will AE be $=$ to the arc AB, very nearly.



For, by ſimilar triangles, $GD:DB::GA:AE=\frac{AG\times BD}{DG}=$

$$\frac{AP+PG}{AP+PG-AD}\times DB= \text{(by the conſt.)}\,\frac{\frac{1}{2}AP+\frac{19AK-21AP}{10AK}\times AD}{\frac{1}{2}AP+\frac{9AK-21AP}{10AK}\times AD}\times DB,$$

which is the arc, nearly, by the above demonſtration.

And this proves the truth of the conſtruction given by *Sir I. Newton*, in his letter of June 13, 1676, to Mr *Oldenburg*. But Mr *Jones*, in publiſhing this letter, hath printed AP inſtead of AD.

COROLLARY II.

When the ellipſe becomes a circle, then $AK=AP$, and the above rule will become $\frac{15AP-2AD}{15AP-12AD}\times DB$ for the length of the circular arc whoſe diameter is AP, verſed ſine AD, and right ſine DB. And this is nearer the truth than any of the other approximations before given for the circular arc, except that in rule 6 for that purpoſe, for the example in that rule calculated by this approximation, gives $6\cdot1170598$ for the length of the arc.

Then $x\times\dfrac{15p+\dfrac{19C-21p}{C}y}{15p+\dfrac{9C-21p}{C}y}=3\times\dfrac{15\times32+\dfrac{19\times18-21\times32}{18}y}{15\times32+\dfrac{9\times18-20\times32}{18}y}$

$=3\times\dfrac{18\times15\times32+\overline{19\times18-21\times32}\times y}{18\times15\times32+\overline{9\times18-21\times32}\times y}=3\times\dfrac{3\times15\times32+\overline{19\times3-7\times16}\times{}^{\cdot}285787471 5}{3\times15\times32+\overline{9\times3-7\times16}\times{}^{\cdot}2857874715}$

$=3\times1{\cdot}006056=3{\cdot}018168=$ the length of the arc very nearly.

PROBLEM V.

** To find the area of an ellipſe.*

RULE I.

Multiply continually together the two axes and the number ·7854 for the area of the ellipſe.

That is, $ntc=$ the area, putting $t=$ the tranſverſe, $c=$ the conjugate, and $n={}^{\cdot}7854$. By corollary 4.

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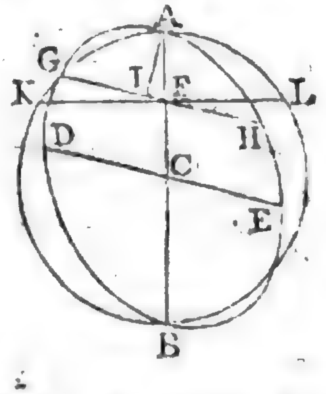
* GENERAL INVESTIGATION.



Let AB, DE, be any two conjugate diameters of the ellipſe $ADBEA$, GH a double ordinate to the diameter AB, $AKBL$ a circle whoſe diameter is AB, KL a double ordinate of the circle to the diameter AB, and AI perpendicular to GH.

Now whilſt KL by flowing generates the circular ſegment KAL, GH will deſcribe the elliptic ſegment GAH; but the velocity of KL is to that of GH as FA to AI; and by the property of all conjugate diameters, as $AC:CD::$($\sqrt{AF\times FB}=$, by the nature of the circle,) $KF:FG$; wherefore as the circular ſegment KAL is to the elliptic ſegment GAH, ſo is $AC\times AF$ to $CD\times AI$, ſo is $AC\times$ radius to $CD\times$ ſine $\angle IFA$, ſo is AC to $CD\times$ ſine $\angle C$, putting 1 for the radius.

EXAMPLE.

If the axes of an ellipſe be 35 and 25, what is the area.

$\cdot 7854 \times 35 \times 25 = 687 \cdot 225$ = the area required.

RULE

COROLLARY I.

The whole ellipſe and circle, deſcribed upon any diameter of it, are to each other as the correſponding ſegments cut off them by their particular double ordinates *GH*, *KL*, paſſing through the ſame point of that diameter.

COROLLARY II.

As radius : ſine of the angle made by any two conjugate diameters :: a mean proportional between the two circles deſcribed upon thoſe diameters : the ellipſe

For $\left\{\begin{matrix} r \times d : s \times c :: \odot d : \bigcirc \\ r \times c : s \times d :: \odot c : \bigcirc \end{matrix}\right\}$ by cor. 1.

Hence ($r^2 cd : s^2 cd$ ::) $r^2 : s^2 :: \odot d \times \odot c : \bigcirc^2$,

And $r : s :: \sqrt{\odot d \times \odot c} : \bigcirc$.

Where d and c are the diameters, s the ſine of the $\angle$ made by them to the radius r, $\odot$ denotes a circle, and $\bigcirc$ an ellipſe.

COROLLARY III.

If the conjugate diameters be equal to each other, it will follow that, As radius : to the ſine of the angle made by the equal conjugate diameters :: ſo is the circle deſcribed on one of thoſe diameters : to the ellipſe.

COROLLARY IV.

The ellipſe is a mean proportional between the two circles deſcribed on the two axes.

For they make with each other a right angle, whoſe ſine is = to the radius.

COROLLARY V.

As radius : to the ſine of the angle made by any two conjugate diameters :: ſo is the circle whoſe diameter is a mean proportional between the conjugate diameters : to the ellipſe.

This follows from corollary 2.

COROLLARY VI.

The ellipſe is equal to a circle whoſe diameter is a mean proportional between the two axes.

From corollary 4.

Co-

RULE II.

Multiply continually together any two conjugate diameters, the natural ſine of their included angle, and the number ·7854.

That is, $dcns$ = the area, putting d and c = any two conjugate diameters, s = ſine of their included ∠, and n = ·7854.

Ex-

COROLLARY VII.

As an ellipſe : is to the rectangle of its two axes (or to the rectangle of any two conjugate diameters drawn into the ſine of their included ∠, the radius being 1) :: ſo is any circle : to the ſquare of its diameter.—Any two like ſegments or zones of the ellipſe and circle are alſo in the ſame proportion.

COROLLARY VIII.

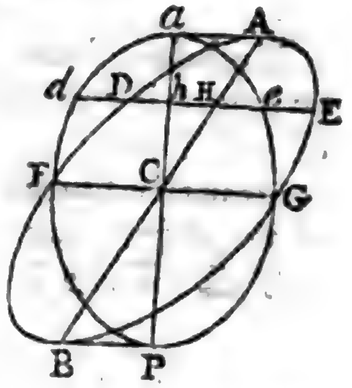
Ellipſes, and their like ſegments, are to one another as the rectangles of their axes, or as the rectangles of any conjugate diameters forming the ſame angle in each.

COROLLARY IX.

Similar ellipſes are to one another as the ſquares of their like diameters.

COROLLARY X.

From corollary 7 comes alſo the following conſtruction.



Let ADE be an oblique ſegment of the ellipſe $AFBGA$, cut off by a double ordinate to the diameter AB, FG being the conjugate. Through the center C draw aP perpendicular to FG and meeting, in a, P, with Aa, BP, both parallel to FG; then about the axes aP, FG, deſcribe the ellipſe $aFPGa$, meeting the ordinate, produced, in e and d. Then will the right elliptic ſegment dae be equal to the oblique ſegment DAE, as well as the whole ellipſe $aFPGa$ equal to the ellipſe $AFBGA$; moreover their correſponding ordinates de, DE, parallel to the common diameter FG, are every where equal, as are the like parts or zones contained between any two of ſuch ordinates. And the ſame may be ſaid of all ellipſes contained between the parallels Aa, BP, infinitely produced : in which property they reſemble parallelograms of the ſame baſe and between the ſame parallels.

EXAMPLE.

If two conjugate diameters of an ellipſe be 28 and 32, and their included angle 77° 34½′; required its area.

The ſine of 77° 34½′ is ·9765625; wherefore ·9765625 × 32 × 28 × ·7854 = 687·225 = the area.

PROBLEM VI.

To find the area of the ſegment of an ellipſe cut off by a double ordinate to either axe, that is, by a line perpendicular to that axe.

RULE I.

Find the correſponding ſegment of the circle deſcribed upon the ſame axe to which the cutting line or baſe of the ſegment is perpendicular. Then as this axe : is to the other axe :: ſo is the circular ſegment : to the elliptic ſegment.

This follows from corollary 7 to the laſt problem.

RULE II.

Find the tabular circular ſegment whoſe verſed ſine or height is equal to the quotient of the height of the elliptic ſegment divided by its axe: Then multiply continually together this ſegment and the two axes of the ellipſe, for the area of the ſegment required.

This follows from the former.

EXAMPLE.

What is the area of an elliptic ſegment cut off by a line parallel to and at the diſtance of 7½ from the leſs axe, the axes being 35 and 25.

Here 17½ − 7½ = 10 = the height of the ſegment.

And $10 \div 35 = 2 \div 7 = \cdot 2857\frac{1}{7}$ = the tabular verſed ſine, whoſe ſegment is $\cdot 1851669$.

Then $\cdot 1851669 \times 35 \times 25 = 162 \cdot 0210375$ = the area of the leſs ſegment.

If the greater ſegment had been required. Then $\cdot 785398 16 - \cdot 1851669 = \cdot 60023126$. And $\cdot 60023126 \times 25 \times 35 = 525 \cdot 2023525$ = the area of the greater ſegment.

EXAMPLE II.

What is the area of the elliptic ſegment cut off by a double ordinate perpendicular to the conjugate axe at the diſtance of $7\frac{1}{2}$ from the center, the axes being 35 and 25.

Here $12\frac{1}{2} - 7\frac{1}{2} = 5$ = the altitude of the ſegment.

And $5 \div 25 = 1 \div 5 = \cdot 2$ = the tabular verſed ſine, whoſe correſponding ſegment is $\cdot 1118238$.

Hence $\cdot 1118238 \times 25 \times 35 = 97 \cdot 845825$ = the area of the leſs ſegment.

Again, $\cdot 78539816 - \cdot 1118238 = \cdot 67357436$. And $\cdot 67357436 \times 25 \times 35 = 589 \cdot 377565$ = the greater ſegment.

PROBLEM VII.

To find the area of an elliptic ſegment cut off by a double ordinate to any diameter; that is, by a line oblique to the axes.

RULE.

Divide the abſciſſa *AF* (fig. to prob. 5) of the double ordinate by its diameter *AB*, and find the tabular circular ſegment whoſe verſed ſine is the quotient: Then multiply continually together the tabular area and the two axes. Or the tabular area, the diameter AB to which the baſe of the ſegment is a double ordinate, its conjugate diameter GH, and the ſine of their included angle, for the area of the elliptic ſegment required.[*]

EXAMPLE.

The axes of an ellipſe being 35 and 25, it is required to find the area of a ſegment whoſe baſe is a double ordinate to a diameter whoſe length is 33, it being divided by the double ordinate into the two abſciſſas 7 and 26.

Here $FA \div AB = 7 \div 33 = \cdot 2121\frac{7}{33}$ = the tabular verſed ſine; to which correſponds the area $\cdot 12162869$.

Hence $\cdot 12162869 \times 25 \times 35 = 106 \cdot 425104$ = the ſegment GAH.

Moreover, $\cdot 78539816 - \cdot 12162869 = \cdot 66376947$. And $\cdot 66376947 \times 25 \times 35 = 580 \cdot 79828625$ = the greater ſegment GBH.

PRO-

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[*] DEMONSTRATION.

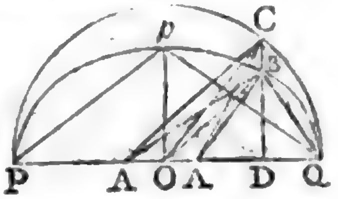
For, by cor. 7, prob. 5, as AB^2 : to circular ſegment LAK :: ſo is ($AB \times DE \times s.LC =$) rectangle of the two axes : to elliptic ſegment GAH; but circular ſegment $LAK = AB^2 \times$ tabular circular ſegment (s) whoſe verſed ſine is $\frac{FA}{AB}$; therefore ($AB^2 : AB^2 \times s$ ::) $1 : s$ :: rectangle of the axes to elliptic ſegment. Q.E.D.

PROBLEM VIII.

To find the trilineal area ABQ *included by either axe, a line drawn from any point in it, and their intercepted arc.*

RULE.

Draw the ordinate DB meeting the circle deſcribed upon the ſaid axe in C, and draw AC, OC.



Then, As the above-ſaid axe:
Is to the other axe::
So is the circular trilineal ACQ:
To the elliptic trilineal ABQ.*

Note.

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* DEMONSTRATION.

For, by the inveſtigation and corollaries to prob. 5, As the axe A : to the axe a $::CD : DB ::$ circular ſegment QCD : elliptic ſegment QBD :: (becauſe of the common baſe AD) triangle ACD : triangle ABD :: (by compoſition) the trilineal ACQ : trilineal ABQ, *Q.E.D.*

COROLLARY I.

Since, by rule 2 prob. 6 ſect. 1 part 2, the circular arc QC (putting $CD=y$, and $OQ=r$) is $=y\times : 1+\frac{y^2}{3.2r^2}+\frac{3y^4}{5.2.4r^4}+\frac{3.5y^6}{7.2.4.6r^6}$ &c. we ſhall have $\frac{1}{2}ry\times : 1+\frac{y^2}{3.2r^2}+\frac{3y^4}{5.2.4r^4}+\frac{3.5y^6}{7.2.4.6r^6}$ &c. $=$ the ſector OCQ, which being increaſed or diminiſhed by $\frac{1}{2}OA\times DC=$ the triangle ACO, we ſhall have $\frac{1}{2}y\times : AQ+\frac{y^2}{3.2r}+\frac{3y^4}{5.2.4r^3}+\frac{3.5y^6}{7.2.4.6r^5}$ &c. for the general value of the trilineal ACQ; and conſequently $\frac{cy}{2r}\times : AQ+\frac{y^2}{3.2r}+\frac{3y^4}{5.2.4r^3}+\frac{3.5y^6}{7.2.4.6r^5}$ &c. $=$ the elliptic trilineal ABQ, r being the radius of the circle, or ſemi-axe OQ of the ellipſe; and c the other ſemi-axe.

And ſince $r : c :: y : BD = z$, we ſhall have $y=\frac{rz}{c}$, and conſequently the value of the elliptic trilineal ABQ, expreſſed in terms of its own ordinate

Note. It is evident that the triangle *ACO* added to or ſubtracted from the circular ſector *OCQ*, will give the circular trilineal *ACQ* according as *AQ* is greater or leſs than half the axe. Or the ſemi-ſegment *CDQ* increaſed or diminiſhed by the triangle *ACD* will give the ſame.

Ex-

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ordinate z (BD) and ſemi-axes r, c, will be $\frac{1}{2} z \times : AQ + \frac{rz^2}{3.2c^2} + \frac{3rz^4}{5.2.4c^4} + \frac{3.5rz^6}{7.2.4.6c^6}$ &c.

COROLLARY II.

If A coincide with O, the triangle AOB will vaniſh, and the trilineal become barely a ſector OBQ, whoſe value will be $\frac{1}{2} rz \times : 1 + \frac{z^2}{3.2c^2} + \frac{3z^4}{5.2.4c^4} + \frac{5.3z^6}{7.2.4.6c^6}$ &c.

COROLLARY III.

When $z = c$, then the quadrant OpQ will be $\frac{1}{2} rc \times : 1 + \frac{1}{3.2} + \frac{3}{5.2.4} + \frac{5.3}{7.2.4.6}$ &c.

COROLLARY IV.

If A coincide with P, then $AQ = QP = 2r$, and the trilineal PBQ will be $= \frac{1}{2} rz \times : 2 + \frac{z^2}{3.2c^2} + \frac{3z^4}{5.2.4c^4} + \frac{5.3z^6}{7.2.4.6c^6}$ &c.

And when B coincides with p, then will the trilineal PpQ be $= \frac{1}{2} rc \times : 2 + \frac{1}{3.2} + \frac{3}{5.2.4} + \frac{5.3}{7.2.4.6}$ &c.

COROLLARY V.

If from the double of the quadrant in corollary 3, be taken the trilineal in corol. 4, there will remain $r \times : \frac{c-z}{1} + \frac{2c^3-z^3}{6.2c^2} + 3\frac{2c^5-z^5}{10.2.4c^4} + 5.3\frac{2c^7-z^7}{14.2.4.6c^6}$ &c. for the value of the ſegment PpB.

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EXAMPLE.

Given the axes 35 and 25 of an ellipſe, and an ordinate to the tranſverſe 10; required the area included by the tranſverſe and a line drawn from the

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COROLLARY VI.

If from the value of the ſector OBQ in corollary 2, be taken that of the triangle $OBQ = \frac{1}{2}rz$, there will remain $\frac{1}{2}rz \times : \frac{z^2}{3.2c^2} + \frac{3z^4}{5.2.4c^4} + \frac{5.3z^6}{7.2.4.6c^6}$ &c. for the value of the ſegment BQ.

COROLLARY VII.

When B coincides with p, the ſeries in each of the two laſt corollaries becomes $\frac{1}{2}rc \times : \frac{1}{3.2} + \frac{3}{5.2.4} + \frac{5.3}{7.2.4.6}$ &c. for the value of each of the ſegments Pp, pQ.

COROLLARY VIII.

If A be the focus of the ellipſe, then $AQ = r \pm \sqrt{rr - cc}$, and the trilineal $ABQ = \frac{1}{2}z \times : r \pm \sqrt{rr - cc} + \frac{rz^2}{3.2c^2} + \frac{3rz^4}{5.2.4c^4} + \frac{3.5rz^6}{7.2.4.6c^6}$ &c. $= \frac{1}{2}rz \times : 1 \pm \sqrt{1 - \frac{cc}{rr}} + \frac{z^2}{2.3c^2} + \frac{3z^4}{2.4.5c^4} + \frac{3.5z^6}{2.4.6.7c^6}$ &c.

COROLLARY IX.

Putting $T = ABQ = \frac{1}{2}rz \times : a + \frac{z^2}{2.3c^2} + \frac{3z^4}{2.4.5c^4}$ &c. (a being $= 1 \pm \sqrt{1 - \frac{cc}{rr}}$) and $q = \frac{2T}{r} = z \times : a + \frac{z^2}{2.3c^2} + \frac{3z^4}{2.4.5c^4}$ &c. we ſhall have, by reverting the ſeries, $z = \frac{q}{a} - \frac{q^3}{6c^2a^4} + \frac{10 - 9a}{120c^4a^7}q^5 - \frac{280 - 504a + 225a^2}{5040c^6a^{10}}q^7$ &c. $= \frac{2T}{r} \times : \frac{1}{a} - \frac{1}{6a^4}p^2 + \frac{10 - 9a}{120a^7}p^4 - \frac{280 - 504a + 225a^2}{5040a^{10}}p^6$ &c. where $p = \frac{2T}{rc}$; which ſeries generally converges very faſt, and affords a good direct ſolution of *Kepler*'s problem.

Suppoſe, for example, it were required to find the true anomaly of the earth, and its diſtance from the ſun, anſwering to the mean anomaly top of the ordinate to cut the tranſverſe in an angle of 40 degrees.

Here, as $pq = 25 : PQ = 35 :: DB = 10 : DC = 14$.

As rad. : tang. $\angle B(50^\circ) :: BD : DA = 11{\cdot}917536$.

Hence $AD \times \frac{1}{2} DC = 83{\cdot}422752 =$ the $\triangle\, ADC$.

As $25 : 35 :: \sqrt{12{\cdot}5^2 - 10^2} : 10{\cdot}5 = OD$, and hence $QO - OD = 7 = DQ$. Then, by the table of circular ſegments, the ſemi-ſegment CDQ will come out $68{\cdot}4920775$.

The triangle ADC + ſegment $CDQ = 151{\cdot}9143395$ = the trilineal CAQ.

Wherefore as $35 : 25 :: 151{\cdot}9143395 : 108{\cdot}5102425$ = the area of the elliptic trilineal BAQ required.

PRO-

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anomaly of 1 ſign or 30 degrees, the excentricity being ·01681 to the mean diſtance 1.

Here $PO = r = 1$, $AO = {\cdot}01681$, $Op = c = \sqrt{1 - {\cdot}01681^2} = {\cdot}9998587$, the area of a quadrant of the ellipſe $= {\cdot}785398$ &c. $\times\, rc = {\cdot}7852872$; then, as 3 ſigns : 1 ſign :: $\cdot 7852872 : {\cdot}2617624 = T$; hence $\frac{2T}{r} = 2T = {\cdot}5235248$, and $\frac{2T}{rc} = \frac{2T}{c} = \frac{2}{3} \times {\cdot}78539$ &c. $= {\cdot}5235988 = p$: alſo $a = 1{\cdot}01681$, and $\frac{p^2}{a^3} = {\cdot}2589869$.

Then, 1ſt term $A = \frac{1}{a} =$ — — — $+{\cdot}98346790$

2d — $B = \frac{1}{6} A \times \frac{p^2}{a^3} =$ — — $-{\cdot}04245088$

3d — $C = \frac{10 - 9a}{20} B \times \frac{p^2}{a^3} =$ — $+{\cdot}00046655$

4th — $D = \frac{280 - 504a + 225a^2}{42} C \times \frac{p^2}{a^3} =$ $-{\cdot}00000044$

the ſum = $\overline{{\cdot}94148313}$

which drawn into $\frac{2T}{r} = 2T = {\cdot}5235248$, gives $\cdot 4928897 = DB = x$.

But

PROBLEM IX.

To find the ſurface of a ſpheroid.

*RULE I.

For both ſpheroids.

Let f denote the fixed axe, or the axe about which the ellipſe is conceived to revolve, r the revolving axe; and put $p=3{\cdot}14159$ &c. $q=\frac{ff\backsim rr}{ff}$: Then will the ſurface (s) of the ſpheroid be expreſſed by either of the following ſerieſes, uſing the upper or under

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But $OD=\frac{r}{c}\sqrt{cc-zz}={\cdot}87005233$; hence $DA=AO+OD={\cdot}88686233$, and as $AD:DB$ :: rad. : tang. $\angle BAD=29{\cdot}063904^\circ=29^\circ\,3'\,50''\,3'''$ the true anomaly; alſo, as rad. : ſec. $\angle BAD :: DA:AB=1{\cdot}0146255$, the diſtance of the earth from the ſun to the mean diſtance 1.

* DEMONSTRATION.

Put the fixed ſemi-axe $Tc=cR=a$ (fig. to prob. 4), $Cc=cO=b$, $cA=x$, $AB=y$, arc $CB=z$, and $3{\cdot}14159$ &c. $=p$. Then, $a:b::\sqrt{aa-xx}:\frac{b}{a}\sqrt{aa-xx}=y$; hence $\dot{y}=-\frac{bx\dot{x}}{a\sqrt{aa-xx}}$, and $\dot{z}=\sqrt{\dot{x}^2+\dot{y}^2}=\sqrt{\dot{x}^2+\frac{b^2x^2\dot{x}^2}{a^2\times\overline{a^2-x^2}}}=\frac{\dot{x}}{a}\sqrt{\frac{a^4-x^2\times\overline{aa-bb}}{aa-xx}}$; and conſequently the fluxion of the ſurface $\dot{s}=2py\dot{z}=\frac{2pb\dot{x}}{aa}\sqrt{a^4-x^2\times\overline{aa-bb}}=\frac{2pb\dot{x}}{aa}\sqrt{a^4-c^2x^2}$ (putting $cc=aa-bb$) $=\frac{2pbc\dot{x}}{aa}\sqrt{\frac{a^4}{cc}-xx}=\frac{2pbc\dot{x}}{aa}\times:\frac{a^2}{c}-\frac{cx^2}{2a^2}-\frac{c^3x^4}{2.4a^6}-\frac{3c^5x^6}{2.4.6a^{10}}$ &c. and the fluent is $s=2pbx\times:1-\frac{c^2x^2}{2.3a^4}-\frac{c^4x^4}{2.4.5a^8}-\frac{3c^6x^6}{2.4.6.7a^{12}}-\frac{3.5c^8x^8}{2.4.6.8.9a^{16}}$ &c. and if for $\frac{cc}{aa}$ be under ſigns according as it is an oblong or oblate ſpheroid, viz. $s = prf \times : 1 \mp \frac{1}{2.3} q - \frac{1}{2.4.5} q^2 \mp \frac{3}{2.4.6.7} q^3 - \frac{3.5}{2.4.6.8.9} q^4$ &c. or $prf \times : 1 \mp \frac{A}{2.3} q - \frac{3B}{4.5} q \mp \frac{3.5C}{6.7} q - \frac{5.7D}{8.9} q$ &c. where *A*, *B*, *C*, &c. are the ſeveral preceding terms.

EXAMPLE.

If the axes be 50 and 40, required the ſurface of each ſpheroid.

I. *For the oblong ſpheroid.*

Here $f = 50$, $r = 40$, and $\frac{ff - rr}{ff} = \frac{30 \times 30}{50 \times 50} = \frac{9}{25} = {}^{\cdot}36 = q$.

Hence

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be written *dd*, the ſeries will become $s = 2pbx \times : 1 - \frac{d^2x^2}{2.3a^2} - \frac{d^4x^4}{2.4.5a^4} - \frac{3d^6x^6}{2.4.6.7a^6} - \frac{3.5d^8x^8}{2.4.6.8.9a^8} =$ (when $x = a$) $2pab \times : 1 - \frac{d^2}{2.3} - \frac{d^4}{2.4.5} - \frac{3d^6}{2.4.6.7} - \frac{3.5d^8}{2.4.6.8.9}$ &c. $= 2pab \times 1 - \frac{A}{2.3} d^2 - \frac{3B}{4.5} d^2 - \frac{3.5C}{6.7} d^2 - \frac{5.7D}{8.9} d^2$ &c. where, when *cc* ($= aa - bb$) is affirmative, viz. in the caſe of an oblong ſpheroid, all the ſigns after the firſt will be negative; but for an oblate ſpheroid, *cc* being negative, the ſigns after the firſt will become alternately *plus* and *minus*. *Q. E. D.*

COROLLARY I.

The ſeries above will not always converge for an oblate ſpheroid, viz. when *d* is greater than 1, or when $\frac{bb}{aa}$ is greater than 2, the ſeries will not converge.

COROLLARY II.

Hence the ſurfaces of ſimilar ſpheroids, and alſo of their like parts, are to each other as the rectangles of their axes, or as the ſquares of the like dimenſions.

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Hence 1ſt term $A =$ — — $1\cdot$

2d term $B = \frac{A}{2.3}q = -0\cdot06$

3d term $C = \frac{3B}{4.5}q = -0\cdot00324$

4th term $D = \frac{3.5C}{6.7}q = -0\cdot0004166-$

5th term $E = \frac{5.7D}{8.9}q = -0\cdot000072900$

6th term $F = \frac{7.9E}{10.11}q = -0\cdot0000150+$

7th term $G = \frac{9.11F}{12.13}q = -0\cdot0000035-$

8th term $H = \frac{11.13G}{14.15}q = -0\cdot0000009-$

the ſum of the negative terms $= -0\cdot0637489$, which taken from the firſt term 1, leaves $\cdot936251$ for the ſum of the ſeries; and this being drawn into $prf = 3\cdot14159265 \times 50 \times 40 = 6283\cdot18531$, will produce $5882\cdot6385$ for the ſurface required.

II. *For the oblate ſpheroid.*

Here $f = 40$, $r = 50$, and $\frac{rr-ff}{ff} = \frac{30\times30}{40\times40} = \frac{9}{16} = q$. Then by increaſing the values of the terms, found above for the oblong ſpheroid, in proportion to to ſuch power of the ratio of the value of q in this caſe to its value in the former, whoſe exponent is reſpectively 1 leſs than the number of each term; viz. in the proportion of the 0, 1, 2, 3, &c. power of $\frac{9}{25}$ to $\frac{9}{16}$, or of 16 to 25, or of 64 to 100; we ſhall have

$$\begin{aligned}
\text{the 1ſt term} &= 1 \times \overline{\tfrac{100}{64}}\Big|^{0} = 1 \times 1 &&= +1\cdot \\
\text{2d term} &= \cdot06 \times \tfrac{100}{64} &&= +0\cdot09375 \\
\text{3d term} &= \cdot00324 \times \overline{\tfrac{100}{64}}\Big|^{2} &&= -0\cdot00791 \\
\text{4th term} &= \cdot0004166 \times \overline{\tfrac{100}{64}}\Big|^{3} &&= +0\cdot00159 \\
\text{5th term} &= \cdot0000729 \times \overline{\tfrac{100}{64}}\Big|^{4} &&= -0\cdot00044 \\
\text{6th term} &= \cdot0000150 \times \overline{\tfrac{100}{64}}\Big|^{5} &&= +0\cdot00014 \\
\text{7th term} &= \cdot0000035 \times \overline{\tfrac{100}{64}}\Big|^{6} &&= -0\cdot00005 \\
\text{8th term} &= \cdot0000009 \times \overline{\tfrac{100}{64}}\Big|^{7} &&= +0\cdot00002
\end{aligned}$$

ſum of the affirmative terms $+1\cdot09550$
ſum of the negative terms $-0\cdot00840$
the difference 1.0871,

then $6283\cdot18531 \times 1\cdot0871 = 6830\cdot4507$ = the ſurface of the oblate ſpheroid required.

RULE II. *For both ſpheroids.*

Divide the revolving axe by the fixed axe, and call the quotient q. Find the difference between 1 and the ſquare of q, and call the ſquare root of that difference s.

For an oblong ſpheroid, multiply $\cdot01745329$ by the degrees in the arc whoſe ſine is s, and call the product P.

For an oblate ſpheroid, multiply $2\cdot302585$ by the logarithm of the ſum of s and q, and call the product P, alſo.

Multiply P into the fixed axe and divide the product by s; to the quotient add the revolving axe; then the ſum multiplied into $\frac{1}{2}$ the circumference of the greateſt circle, viz. into 3·14159 &c. and into $\frac{1}{2}$ the revolving axe, will give the ſurface required.

That is, putting $f=$ the fixed axe,
$r=$ the revolving axe,
$q=r\div f$,
$s=\sqrt{1 \backsim qq}$,
$p=3{\cdot}14159$ &c.
$P={\cdot}01745329\times$ degrees to the ſine s, in the oblong, and $=$ $2{\cdot}302585\times$ log. of $s+q$ in the oblate ſpheroid;

then $\overline{\frac{Pf}{s}+r}\times\frac{1}{2}pr=$ the ſurface of the ſpheroid.*

EXAMPLE.

The axes of a ſpheroid being 50 and 30, required the ſurface.

I. For the oblong ſpheroid.

Here $f=50$, $r=30$, $q=\frac{30}{50}={\cdot}6$, $s=\sqrt{1-qq}=\sqrt{1-{\cdot}36}=\sqrt{{\cdot}64}={\cdot}8$, which is the ſine of 53·130105 degrees.

Then $53{\cdot}130105\times{\cdot}01745329={\cdot}92729513=P$, and $\frac{50\times{\cdot}92729513}{{\cdot}8}+30\times\frac{30\times3{\cdot}14159\text{ \&c.}}{2}=4144{\cdot}8263=$ the ſurface of the oblong ſpheroid.

II. For

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* DEMONSTRATION.

For the fluent of $\frac{2pb\dot{x}}{aa}\sqrt{a^4-c^2x^2}$ (the fluxion of the ſurface in the demonſtration of the firſt rule) is, when $x=a$, $pb\times\overline{b+\frac{aa}{c}P}$, where $P={\cdot}017453\times$ degrees in the arc whoſe ſine is $\frac{c}{a}$, when a is greater than b; or $P=2{\cdot}3025$ &c. $\times$ log. of $\frac{b+c}{a}$, when a is leſs than b, c being $=\sqrt{aa \backsim bb}$. Q.E.D.

II. For the oblate ſpheroid.

Here $f=30$, $r=50$, $q=\frac{50}{30}=\frac{5}{3}$, and $s=\sqrt{1 \backsim \frac{25}{9}}=\sqrt{\frac{16}{9}}=\frac{4}{3}$; hence $s+q=\frac{5}{3}+\frac{4}{3}=\frac{9}{3}=3$, whoſe log. is $\cdot4771213$; and $P=\cdot4771213\times2{\cdot}302585=1{\cdot}0986124$.

Then $\frac{30\times1{\cdot}0986124}{\frac{4}{3}}+50\times\frac{50\times3{\cdot}14159\ \&c.}{2}=5868{\cdot}39918$ = the ſurface of the oblate ſpheroid.

RULE III.

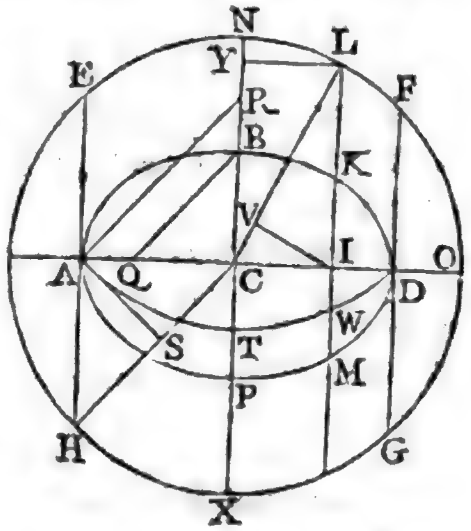
For the oblong ſpheroid.

Find q and s as in the laſt rule.

Find alſo the area of the circular middle zone whoſe radius is half the quotient of the fixed axe divided by s, and the diſtance of each end from the center is half the fixed axe, and call this zone z.

Then multiply continually together z, q, s, and $3{\cdot}14159$ &c. for the ſurface of the oblong ſpheroid.

That is, $pqsz$ = the ſurface of the oblong ſpheroid $ABDP$, where z is equal to the middle zone $EFGH$ of the circle whoſe radius is $\frac{\frac{1}{2}f}{s}$.*



Ex-

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* DEMONSTRATION.

For ſince the fluxion of the ſurface $BKMP$ is $\frac{2pbc\dot{x}}{aa}\sqrt{\frac{a^4}{cc}-xx}$, and

that

EXAMPLE.

Taking again the ſame example,

We have $\frac{1}{2}f \div s = \frac{25}{{\cdot}8} = 31{\cdot}25$ = the radius, or $62{\cdot}5$ = the diameter of the circle.

Hence $OC - CD = 31{\cdot}25 - 25 = 6{\cdot}25 = DO$, and $DO \div 2OC = 6{\cdot}25 \div 62{\cdot}5 = {\cdot}1$ = the tabular verſed ſine, to which belongs the area $\cdot04087528$; and this taken from $\cdot39269908$, the tabular ſemi-circle, leaves $\cdot3518238$ for half the tabular zone.

Whence $\cdot3518238 \times 2 \times 62{\cdot}5 \times 62.5 = 2748{\cdot}6234375$ = the middle zone $EFGH = z$.

Then $pqsz = 3{\cdot}14159$ &c. $\times {\cdot}6 \times {\cdot}8 \times 2748{\cdot}6234375 = 4144{\cdot}8265$ = the ſurface, nearly the ſame as before.

RULE IV.

For the oblong ſpheroid.

Between the ſemi-conjugate BC (ſee the figure to the laſt rule) and the ſum of the conjugate BP and circular arc EF, find a mean proportional, and it will be the radius of a circle equal to the ſurface of the ſpheroid; the circle being deſcribed as in the laſt rule.

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that of the circular area $NLIC = \dot{x}\sqrt{\frac{a^4}{cc} - xx}$, which being to the former in the conſtant ratio of 1 to $\frac{2pbc}{aa}$, their fluents muſt be continually in the ſame ratio; and conſequently the ſurface $BKMP = \frac{2pbc}{aa} \times CNLI = pqs \times 2CNLI = p \times \frac{BC}{CN} \times 2CNLI$. And the whole ſurface $ABDPA = pqs \times EFGH$. Q. E. D.

That is, multiply continually together BC, $BP+EF$, and 3·14159 &c. for the ſurface.*

Ex-

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* DEMONSTRATION.

Apply $BQ = AC$, and draw AR parallel to BQ; then is AR = the radius CN; for, by the conſtruction, CN is $= \frac{aa}{\sqrt{aa-bb}} = \frac{AC^2}{\sqrt{QB^2-BC^2}} = \frac{AC^2}{CQ} = \frac{AC\times BQ}{QC} =$ (by ſimilar triangles) AR. Draw CH, and AS perpendicular to it.

Then, by the laſt rule, the ſurface BAP is $= p\times\frac{BC}{CN}\times 2CNEA$ or $2CXHA = p\times BC\times\frac{2\text{ ſector }CXH+2\Delta HAC}{CH} = p\times BC\times\overline{NE+AS}$.

But, by ſimilar triangles, $\left\{\begin{matrix} HC : CA :: AH : AS \\ AR : BQ :: RC : CB \end{matrix}\right\}$; and ſince $CH = AR$, and $CA = BQ$, by the conſtruction; as alſo $AH = RC$, by reaſon of the equal hypothenuſes CH, AR, and common baſe AC; the fourth terms AS, CB, muſt, alſo, be equal to each other. And conſequently the ſurface $BAP = p\times BC\times\overline{NE+BC} = p\times BC\times\overline{AT+BC}$; and that of the whole ſpheroid $= p\times BC\times\overline{ATD+BP} = p\times BC\times\overline{ENF+BP}$, the arc ATD being deſcribed with the center R and radius RA; viz. equal to a circle whoſe radius is a mean proportional between BC and $ATD+BP$. Q.E.D.

SCHOLIUM.

This conſtruction, for finding a circle equal to the ſurface of an oblong ſpheroid, was firſt given by Mr *Huygens* in his *Horologium Oſcillatorium*, *prop.* 9, but without demonſtration.—The conſtruction is here rendered general for any part or zone $BKMP$ of the ſpheroid; viz. that its curve ſurface is equal to a circle whoſe radius is a mean proportional between BC and $\overline{NL+IV}$ or $\overline{TW+IV}$, IV being perpendicular to CL. For in the ſame manner as $p\times BC\times\overline{NE+AS}$ was found for the ſurface BAP, may the ſurface of any zone $BKMP$ be found to be $p\times BC\times\overline{NL+IV}$.

In imitation of the two rules above for expreſſing the ſurface of the oblong ſpheroid, by means of a circular arc and a circular or elliptic area, I ſhall here ſhew the inveſtigation of others for that of

EXAMPLE.

Taking ſtill the ſame example,

We have the radius $CN = \frac{\frac{1}{2}f}{s} = \frac{25}{\cdot 8} = 31\frac{1}{4}$; then as $31\frac{1}{4}$: $25 = CA$ the ſine of the arc EN :: $1 : \frac{4}{5} = \cdot 8 =$ the ſine of

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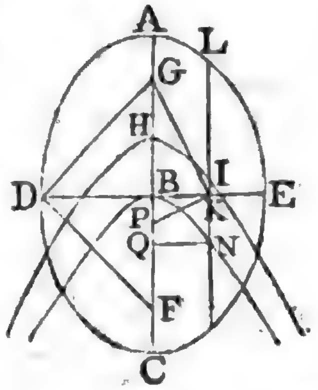
of the oblate ſpheroid, by means of the parabolic arc or hyperbolic area.

Thus if x be $=$ the abſciſſa, and $y =$ the ordinate of a parabola whoſe parameter is n; ſince $x = \frac{yy}{n}$, and $\dot{x} = \frac{2y\dot{y}}{n}$, the fluxion of the arc will be $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\frac{4y^2\dot{y}^2}{nn} + \dot{y}^2} = \frac{\dot{y}}{n}\sqrt{nn + 4yy} = \frac{\dot{y}}{m}\sqrt{mm + yy}$ (putting $m = \frac{1}{2}n = \frac{1}{2}$ the parameter); which being to that of the oblate ſurface, viz. $\frac{2pb\dot{x}}{aa}\sqrt{a^4 + c^2x^2} = \frac{2pbc\dot{x}}{aa}\sqrt{\frac{a^4}{cc} + x^2} = \frac{2pb\dot{x}}{m}\sqrt{mm + xx}$ (putting $m = \frac{a^2}{c}$, and ſuppoſing $x = y$) in the conſtant ratio of 1 to $2pb$, their fluents muſt be in the ſame ratio; and conſequently if P be put to denote the parabolic arc, then will $2pbP$ expreſs the value of the oblate ſurface, which therefore is equal to a circle whoſe radius is a mean proportional between b and $2P$, or between $2b$ and P. And hence this conſtruction. Let DE be the leſs or fixed axe of the ellipſe, AC the greater axe, and F the focus; join D, F, and draw DG perpendicular to DF; biſect BG in H; with the vertex H and focus B deſcribe a parabola HI: Then draw any ordinate LK to the axe DE meeting with the parabola in I; and the ſurface generated by the arc AL will be equal to a circle whoſe radius is a mean proportional between AC and the parabolic arc HI. Or the ſurface generated by AL will be equal to the curve ſurface of a cylinder whoſe diameter is AC and height HI.



Again,

of the ſimilar arc to the radius 1, to which belong 53·130105 degrees; hence $53{\cdot}130105 \times {\cdot}01745329 \times 31\frac{1}{4} = 28{\cdot}977972 =$ the arc EN; then $BP \times \overline{BC+EN} \times 3{\cdot}14159$ &c. $= 4144{\cdot}8262 =$ the ſurface as before, nearly.

RULE V.

For both ſpheroids.

Find q as in the firſt rule, then to or from 1 add or ſubtract $\frac{1}{3}$ of q according as it is the oblate or oblong ſpheroid, and call the ſquare root of the ſum or difference A. Or $A = \sqrt{1 \pm \frac{1}{3}q}$.*

Mul-

Again, ſince $P = \dfrac{\frac{2by}{m}\sqrt{mm+yy} - A}{b}$ (A being the area of an hyperbola whoſe ſemi-axes are b and m, and y the ordinate, ſee cor. 7 rule 1 prob. 6 ſect. 5), the ſurface generated by AL will be equal to $2p \times \frac{2by}{m}\sqrt{mm+yy} - A$; and hence this conſtruction is evident. Let LK, produced, meet, in N, with the hyperbola BN whoſe vertex is B, center A, and ſemi-conjugate axe BG; draw GK, and KP perpendicular to it, and draw the ordinate NQ: Then will the difference between the rectangle $AC \times KP$ and the hyerbolic area BNQ be to the ſurface generated by AL, as the radius of a circle is to its circumference.

* For ſince, by rule 1, the value of the ſurface is $S = 4abp \times : 1 \pm \frac{d^2}{2.3} - \frac{d^4}{2.4.5} \pm \frac{3d^6}{2.4.6.7} - \frac{3.5d^8}{2.4.6.8.9}$ &c. and $4abp\sqrt{1 \pm \frac{1}{3}d^2} = 4abp \times : 1 \pm \frac{d^2}{2.3} - \frac{d^4}{2.4.9} \pm \frac{3d^6}{2.4.6.27} - \frac{3.5d^8}{2.4.6.8.81}$ &c. we ſhall have, by taking the latter of theſe quantities from the former, $S - 4abp\sqrt{1 \pm \frac{1}{3}d^2} = -\frac{d^4}{2.5.9} \pm \frac{5d^6}{4.7.27} - \frac{5d^8}{2.8.81}$ &c. and conſequently $S = 4abp\sqrt{1 \pm \frac{1}{3}d^2}$ nearly, which is rule 5.

Multiply the product of the two axes by $3{\cdot}14159$ &c. and call the product P.

Then the product of A and P will be the ſurface nearly.

That is, $pfr\sqrt{1 \pm \frac{1}{3}q}$.

EXAMPLE.

Let there be taken here the ſame example as at rule 1, in which the axes are 50 and 40.

Then, I. *For the oblong ſpheroid.*

Here $q = \frac{9}{25}$, and $\sqrt{1 - \frac{9}{3 \times 25}} = \sqrt{1 - \frac{3}{25}} = \frac{1}{5}\sqrt{22}$, $= {\cdot}93808 = A$. But $3{\cdot}14159$ &c. $\times 50 \times 40 = 6283{\cdot}1853 = P$. Then $A \times P = 6283{\cdot}1853 \times {\cdot}938 = 5893{\cdot}6278 =$ the ſurface nearly.

II. *For the oblate ſpheroid.*

Here $q = \frac{9}{16}$, and $\sqrt{1 + \frac{9}{3 \times 16}} = \sqrt{1 + \frac{3}{16}} = \frac{1}{4}\sqrt{19}$ $= 1{\cdot}089725 = A$. Then $A \times P = 1{\cdot}0897 \times 6283{\cdot}1853 = 6846{\cdot}7869 =$ the ſurface of the oblate ſpheroid nearly.

* RULE VI.

For both ſpheroids.

To or from 1 add or ſubtract, according as it is for the oblate or oblong ſpheroid, $\frac{1}{6}$ of q, and call $\frac{4}{9}$ of the ſum or difference B.

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* Again, by tranſpoſing the two firſt terms of the ſeries in the value of S, we ſhall have $S - 4abp \times \overline{1 \pm \frac{d^2}{6}} = 4abp \times : -\frac{d^4}{2.4.5} \pm \frac{3d^6}{2.4.6.7} - \frac{3.5d^8}{2.4.6.8.9}$ &c. or $\frac{4}{9} \times \overline{S - 4abp \times \overline{1 \pm \frac{d^2}{6}}} = 4abp \times : -\frac{d^4}{2.5.9} \pm \frac{d^6}{4.7.9}$ —

Then multiply P by the difference between A and B, and $\frac{2}{5}$ of the product will be the ſurface nearly.

That is, $\frac{2}{5}P \times \overline{A-B} = \frac{2}{5} \times pfr \times \overline{\sqrt{\overline{1 \pm \frac{1}{3}q}} - \frac{4}{9} \times \overline{1 \pm \frac{1}{6}q}}$ = the ſurface nearly.

EXAMPLE.

Taking the ſame example, we ſhall have

I. *For the oblong ſpheroid.*

$B = \frac{4}{9} \times \overline{1 - \frac{1}{6}q} = \frac{4}{9} \times \overline{1 - \frac{9}{6 \times 25}} = \frac{4}{9} \times \frac{47}{50} = {}^{\cdot}417777$ &c. Then $A - B = {}^{\cdot}93808 - {}^{\cdot}417777 = {}^{\cdot}5203$. And $\frac{2}{5} \times {}^{\cdot}5203 \times 6283{}^{\cdot}1853 = 5884{}^{\cdot}2029$ = the ſurface of the oblong ſpheroid very nearly.

II. *For the oblate ſpheroid.*

$B = \frac{4}{9} \times \overline{1 + \frac{1}{6}q} = \frac{4}{9} \times \overline{1 + \frac{9}{6 \times 16}} = \frac{4}{9} \times \frac{35}{32} = {}^{\cdot}48611$ &c. Then $A - B = 1{}^{\cdot}0897 - {}^{\cdot}4861 = {}^{\cdot}6036$. And $\frac{2}{5} \times {}^{\cdot}6036 \times 6283{}^{\cdot}1853 = 6826{}^{\cdot}6806$ = the ſurface of the oblate ſpheroid nearly.

RULE VII.

For both ſpheroids.

From the ſum or difference of 1 and $\frac{1}{6}$ of q, according as it is for the oblate or oblong ſpheroid,

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$-\frac{5d^8}{4.8.81}$ &c. which being taken from $S - 4abp\sqrt{\overline{1 \pm \frac{1}{3}d^2}} = -\frac{d^4}{2.5.9} \pm \frac{5d^6}{4.7.27} - \frac{5d^8}{2.8.81}$ &c. the remainder above found, leaves $\frac{5}{9}\overline{S - 4abp \times \sqrt{\overline{1 \pm \frac{1}{3}d^2}} - \frac{4}{9} \times \overline{1 \pm \frac{1}{6}d^2}} = \pm\frac{d^6}{2.7.27} - \frac{5d^8}{4.8.81}$ &c. and conſequently $S = \frac{9}{5} \times 4abp \times \overline{\sqrt{\overline{1 \pm \frac{1}{3}dd}} - \frac{4}{9} \times \overline{1 \pm \frac{1}{6}dd}} = \frac{4}{5}abp \times \overline{9\sqrt{\overline{1 \pm \frac{1}{3}dd}} - 4 \times \overline{1 \pm \frac{1}{6}dd}}$ more nearly, which is rule 6.

roid, take $\frac{1}{40}$ of the ſquare of q, and call $\frac{8}{27}$ of the remainder C.*

Then from A ſubtract the ſum of B and C, multiply the remainder by P, and $\frac{27}{7}$ of the product will be the ſurface of the ſpheroid ſtill nearer.

That is, $\frac{27}{7} P \times \overline{A - B - C}$ = the ſurface nearly, where $C = \frac{8}{27} \times \overline{1 \pm \frac{1}{6} q - \frac{1}{40} qq}$, and the value of the other letters as before.

EXAMPLE.

Taking ſtill the ſame example, in which the axes are 40 and 50;

I. *For the oblong ſpheroid.*

Here again $q = \frac{9}{25} = {\cdot}36$, $P = 6283{\cdot}1853$, $A = {\cdot}93808$, $B = {\cdot}417777$ &c. and $C = \frac{8}{27} \times \overline{1 - \frac{1}{6} q - \frac{1}{40} qq} = {\cdot}277558$. Then $\frac{27}{7} P \times \overline{A - B - C} = 6283{\cdot}1853 \times {\cdot}93628 = 5882{\cdot}8206$ = the ſurface of the oblong ſpheroid very near.

II. *For*

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* Farther, by tranſpoſing the firſt three terms of the original ſeries, and reducing the reſult to make the firſt term of it equal to that of the laſt remainder, we ſhall have $\frac{8}{27} \times \overline{S - 4abp \times \overline{1 \pm \frac{1}{6} d^2 - \frac{1}{40} d^4}} = 4abp \times : \pm \frac{d^6}{2.7.27} - \frac{5d^8}{6.8.81}$ &c. which being taken from the laſt remainder, $\frac{5}{9} S - 4abp \times \overline{\sqrt{1 \pm \frac{1}{3} dd} - \frac{4}{9} \times \overline{1 \pm \frac{1}{6} dd}} = \pm \frac{d^6}{2.7.27} - \frac{5d^8}{4.8.81}$ &c. leaves $\frac{7}{27} S - 4abp \times \overline{\sqrt{1 \pm \frac{1}{3} dd} - \frac{4}{9} \times \overline{1 \pm \frac{1}{6} dd} - \frac{8}{27} \times \overline{1 \pm \frac{1}{6} d^2 - \frac{1}{40} d^4}} = -\frac{5d^8}{8.12.81}$ &c. and conſequently $S = \frac{27}{7} \times 4abp \times \overline{\sqrt{1 \pm \frac{1}{3} dd} - \frac{4}{9} \times \overline{1 \pm \frac{1}{6} dd} - \frac{8}{27} \times \overline{1 \pm \frac{1}{6} d^2 - \frac{1}{40} d^4}} = \frac{4}{7} abp \times \overline{27 \sqrt{1 \pm \frac{1}{3} dd} - 12 \times \overline{1 \pm \frac{1}{6} dd} - 8 \times \overline{1 \pm \frac{1}{6} d^2 - \frac{1}{40} d^4}}$, nearer ſtill, and this is rule 7.

II. *For the oblate ſpheroid.*

Here. $q = \frac{9}{16} = {\cdot}5625$, $P = 6283{\cdot}1853$, $A = 1{\cdot}089725$, $B = {\cdot}486111$, and $C = \frac{8}{27} \times \overline{1 + \frac{1}{8}q - \frac{1}{40}qq} = {\cdot}32173$. Then $\frac{27}{7} \times \overline{A - B - C} \times P = 1{\cdot}0872 \times 6283{\cdot}1853 = 6831{\cdot}079 =$ the ſurface of the oblate ſpheroid very nearly.

PROBLEM X.

To find the curve ſurface of the fruſtum of a ſpheroid, contained between two planes cutting the ſpheroid perpendicular to the fixed axe, and one of them paſſing through the center.

RULE I.

For both ſpheroids.

Let f denote the fixed, and r the revolving axe; put $p = 3{\cdot}14159$ &c. and $q = \frac{ff \backsim rr}{ff}$, as in the laſt problem; alſo $h =$ the height cA of the fruſtum (ſee the figure to rule 1 of prob. 4) and $z = \frac{4qhh}{ff}$; then will the value of the ſurface be expreſſed by this ſeries $prh \times : 1 \pm \frac{1}{2.3}z - \frac{1}{2.4.5}z^2 \pm \frac{3}{2.4.6.7}z^3 - \frac{3.5}{2.4.6.8.9}z^4$ &c. or $prh \times : 1 \pm \frac{1}{2.3}Az - \frac{3}{4.5}Bz \pm \frac{3.5}{6.7}Cz - \frac{5.7}{8.9}Dz$ &c. where A, B, C, &c. are the ſeveral preceding terms, and the upper or under ſigns to be uſed according as it is the oblate or oblong ſpheroid.*

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* See the inveſtigation of rule 1 of the laſt problem.

EXAMPLE I.

What is the ſurface of the fruſtum of a ſpheroid whoſe height is 15; the diameter of the greater end being 40, and that of the leſs 32?

Here $CO = 40 =$ the revolving axe, $2AB = 32$, and $BH = cA = 15$; hence $HC = Cc - AB = 4$, and $HO = Cc + AB = 36$; and conſequently, by caſe 3 prob. 2, we have as $\sqrt{CH \times HO} = 12 : HB = 15 :: CO = 40 : TR = \frac{15 \times 40}{12} = 50 =$ the fixed axe, which ſhews that the fruſtum is that of an oblong ſpheroid.

Then, as in the example to rule 1 of the laſt problem, $f = 50$, $r = 40$, $q = \frac{9}{25} = {\cdot}36$. But $b = 15$, and $z = \frac{4qbb}{ff} = \frac{4 \times 9 \times 15 \times 15}{25 \times 50 \times 50} = {\cdot}1296$.

$$\begin{array}{llll}
\text{Hence the 1ſt term} & A = & \text{—} & +1{\cdot} \\
\text{2d} \ \text{—} & B = & \frac{1}{2.3}Az & = -0{\cdot}0216 \\
\text{3d} \ \text{—} & C = & \frac{3}{4.5}Bz & = -0{\cdot}0004199+ \\
\text{4th} \ \text{—} & D = & \frac{3.5}{6.7}Cz & = -0{\cdot}0000195- \\
\text{5th} \ \text{—} & E = & \frac{5.7}{8.9}Dz & = -0{\cdot}0000012+ \\
\text{6th} \ \text{—} & F = & \frac{7.9}{10.11}Ez & = \underline{-0{\cdot}0000001-}
\end{array}$$

the ſum of the negative terms is $-0{\cdot}0220407$,
which taken from the firſt term $1{\cdot}0000000$,
leaves $0{\cdot}9779593$

for the value of the infinite ſeries; which being drawn into $prb = 3{\cdot}14159$ &c. $\times 40 \times 15 = 1884{\cdot}955592$; produces $1843{\cdot}4098512$ for the ſurface required.

EXAMPLE II.

It is required to find the curve ſurface of the fruſtum $2cABC$ of a ſpheroid whoſe height is 16; the diameter of the greater end being 50 and that of the leſs 30.

Here the revolving axe $TR = 50$, $HC = AB = 16$, and $BH = Ac = 15$; hence $AR = Rc - cA = 10$, and $AT = Tc + cA = 40$; then, by caſe 3 prob. 2, we ſhall have as $\sqrt{AR \times AT} = 20 : AB = 16 :: TR = 50 : CO = \frac{50 \times 16}{20} = 40$ = the fixed axe, which indicates the fruſtum to be that of an oblate ſpheroid.

Wherefore $f = 40$, $r = 50$, $q = \frac{9}{16}$, $b = 16$, and $z = \frac{4qbb}{ff} = \frac{4 \times 9 \times 16 \times 16}{16 \times 40 \times 40} = \frac{36}{100} = {}^{\cdot}36$, which being the ſame as the converging quantity q in the example to rule 1 of the laſt prob. the ſeveral terms of the ſeries muſt be the ſame as there found, viz.

the 1ſt term $A = +1$
2d — $B = +0{\cdot}06$
3d — $C = -0{\cdot}00324$
4th — $D = +0{\cdot}0004166-$
5th — $E = -0{\cdot}000072900$
6th — $F = +0{\cdot}0000150+$
7th — $G = -0{\cdot}0000035-$
8th — $H = +0{\cdot}0000009-$
9th — $I = -0{\cdot}0000002+$
ſum of the affir. terms $= +1{\cdot}0604325$
ſum of the neg. terms $= -0{\cdot}0033166$
their difference $=$ $1{\cdot}0571159$ the value of the

the ſeries; which being drawn into $prh =$ 3·14159 &c. × 50 × 16 = 2513·27412287, will produce 2656·822036 for the ſurface of the oblate fruſtum required.

RULE II.

For both ſpheroids.

1. Multiply the ſquare root of the difference of the ſquares of the axes by the height of the fruſtum, divide the product by the ſquare of the fixed axe, and call the double of the quotient q.

That is, $q = \frac{2h\sqrt{ff \backsim rr}}{ff}$, h being the height of the fruſtum, f the fixed, and r the revolving axe.

2. In the oblong ſpheroid, call the ſquare root of the difference between 1 and the ſquare of q, A. But in the oblate ſpheroid, let A be the root of the ſum of 1 and the ſquare of q.

That is, $A = \begin{cases} \sqrt{1 - qq} & \text{in the oblong,} \\ \sqrt{1 + qq} & \text{in the oblate.} \end{cases}$

3. In the oblong ſpheroid, let the product of ·01745329 and the degrees whoſe ſine is q be called B. But in the oblate ſpheroid, let B be the product of 2·30258509 and the logarithm of the ſum of q and the root of the ſum of 1 and the ſquare of q.

That is, $B =$

$$\begin{cases} \cdot 0174 \text{ \&c.} \times \text{degrees whoſe ſine is } q \text{ in the oblong,} \\ 2\cdot 302 \text{ \&c.} \times \text{log. of } q + \sqrt{1+qq} \text{ in the oblate.} \end{cases}$$

4. Divide B by q, to the quotient add A, multiply the ſum by the continual product of the height, the revolving axe, and the number 3·14159 &c. and the half of the laſt product will be the ſurface required.

That is, $\frac{1}{2}prh \times \overline{A + \frac{B}{q}}$ = the ſurface.*

EXAMPLE I.

Let there be taken here the firſt example to the laſt rule, in which f is $= 50$, $r = 40$, and $h = 15$.

Then $\frac{2h\sqrt{ff - rr}}{ff} = \frac{30 \times 30}{50 \times 50} = \frac{9}{25} = \cdot 36 = q$. And $A = \sqrt{1 - qq} = \sqrt{1 - \overline{\frac{9}{25}}|^2} = \frac{4\sqrt{34}}{25} = \cdot 9329523$. Alſo the ſine $(q =)$ ·36 anſwers to 21·1001965 degrees, which being drawn into ·01745329, will produce $\cdot 3682678 = B$.

Hence $\overline{A + \frac{B}{q}} \times \frac{1}{2}prh = \cdot 9779593 \times 1884\cdot 955592 = 1843\cdot 40985 12$ = the ſurface, the ſame as before.

EXAMPLE II.

Let there be taken the oblate fruſtum in the ſecond example to the preceding rule, in which the axes are 50 and 40, and the height 16.

Here then $f = 40$, $r = 50$, and $h = 16$; hence $\frac{2h\sqrt{rr - ff}}{ff} = \frac{32 \times 30}{40 \times 40} = \frac{3}{5} = \cdot 6 = q$; and $\sqrt{1 + qq} = \sqrt{1 + \overline{\frac{3}{5}}|^2} = \frac{1}{5}\sqrt{34} = 1\cdot 16619038 = A$; likewiſe

$q+$

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* For this is the fluent of the fluxion of the ſurface in the inveſtigation of the rules in the laſt problem.

$\overline{q+\sqrt{1+qq}} = 1{\cdot}76619038$, whoſe logarithm is ·24703757, which being drawn into 2·30258509 will produce ·56882503 $= B$.

Then $\overline{A+\frac{B}{q}}\times\frac{1}{2}prb = 1{\cdot}057116\times2513{\cdot}27412287 = 2656{\cdot}822287 =$ the ſurface, nearly the ſame as before.

RULE III.

For the oblong ſpheroid.

1. Divide the ſquare of the fixed axe by the root of the difference of the ſquares of the axes, and call the quotient d.

That is, $d = \frac{ff}{\sqrt{ff-rr}}$.

2. Find the area of the circular zone whoſe diameter is d, and its height from the center equal to the height h of the fruſtum of the ſpheroid, and call the area of that zone z.

3. Then as the diameter of the circle is to the circular zone, ſo is the circumference of the greateſt circle of the ſpheroid to the ſurface of its fruſtum.

That is, as $d : z :: pr : \frac{prz}{d} =$ the ſurface of the fruſtum.*

EXAMPLE.

It is required to find the ſurface of the fruſtum of an oblong ſpheroid whoſe height is 15, the axes being 50 and 40.

Here $\frac{ff}{\sqrt{ff-rr}} = \frac{50\times50}{30} = \frac{250}{3} = 83\frac{1}{3} = d$ the diameter of the circle.

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* This is the ſame with rule 3 to the laſt problem, only expreſſed in other terms.

And, by means of the table of circular areas, the area of the zone will be found to be $\cdot 17603268 \times \overline{\frac{250}{3}}\Big|^2$.

Alſo $pr = 3\cdot 14159$ &c. $\times 40 =$ the greateſt circumference of the ſpheroid.

Then, as $\frac{250}{3} : \cdot 17603268 \times \overline{\frac{250}{3}}\Big|^2 :: 3\cdot 14159$ &c. $\times 40 : \cdot 17603268 \times 3\cdot 14159$ &c. $\times 40 \times \frac{250}{3} = 1843\cdot 4099 =$ the ſurface, nearly the ſame as before.

RULE IV.

For the oblong ſpheroid.

Let $BKMP$ be the fruſtum whoſe ſurface is ſought, [figure to rule 3 of the laſt prob.] NLO a quadrant of the circle mentioned in the laſt problem; draw CL, and IV perpendicular to it.

Then the ſurface of the fruſtum will be equal to the circle whoſe radius is a mean proportional between BC and $NL + IV$. Or the ſurface will be equal to the continual product of $\overline{NL + IV}$, BC, and $3\cdot 14159$ &c.*

EXAMPLE.

Let there be taken here the ſame example as before, in which the axes are 50 and 40, and the height 15.

Then ſince, by the example to the laſt rule, $CL = CN = \frac{125}{3}$, we ſhall have as $CL = \frac{125}{3} : LY = 15 ::$ $1 =$

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* This is proved at rule 4 to the laſt problem.

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$1 =$ radius : $\frac{9}{25} = {·36} =$ the ſine of 21·1001965 degrees, which are thoſe contained in the arc NL; and therefore $·01745329 \times 21·1001965 \times \frac{125}{3} = 15·3444958 = NL$. And, by ſimilar triangles, (ſince $IL = \sqrt{LC^2 - CI^2} = \sqrt{\overline{\frac{125}{3}}|^2 - 15^2} = \frac{20\sqrt{34}}{3} = 38·8730126$) as $CL = \frac{125}{3} : LI :: CI = 15 : IV = 13·9942845$.

And hence $\overline{NL + IV} \times 3·14159$ &c. $\times BC = 29·3387803 \times 3·14159$ &c. $\times 20 = 1843·409938 =$ the ſurface, nearly the ſame as before.

RULE V.

For both ſpheroids.

The 5th rule to the laſt problem will ſerve, if inſtead of q we uſe z, as found in rule 1 of this problem, and h the height inſtead of f the fixed axe.

That is, $prh\sqrt{1 \pm \frac{1}{3}z} =$ the ſurface nearly.

EXAMPLE I.

Taking the ſame example as in the laſt rule, we have $r = 40$, $h = 15$, and z (found in the firſt example to rule 1) $= ·1296$.

Conſequently $prh\sqrt{1 - \frac{1}{3}z} = 1884·955592 \times ·97816 = 1843·7879 =$ the ſurface nearly.

EXAMPLE II.

If we take the oblate fruſtum, whoſe height is 16, and the axes 40 and 50; we ſhall have

$r =$

$r=50$, $b=16$, and z (found in the ſecond example to rule 1) $= {\cdot}36$.

Then $prb\sqrt{1+\frac{1}{3}z} = 2513{\cdot}27412287 \times 1{\cdot}0583 = 2659{\cdot}797 =$ the ſurface nearly.

RULE VI.

For both ſpheroids.

This rule will be the ſame as the 6th rule to the laſt problem, uſing z inſtead of q, and b for f as in the laſt rule.

That is, $\frac{2}{5}P\times\overline{A-B} = \frac{2}{5}prb\times\overline{\sqrt{1\pm\frac{1}{3}z}-\frac{4}{9}\times\overline{1\pm\frac{1}{6}z}}$ = the ſurface nearly.

EXAMPLE I.

Taking the 1ſt example to the laſt rule, we have $P = 1884{\cdot}955592$, $A=\sqrt{1-\frac{1}{3}z} = {\cdot}97816157$, and $B=\frac{4}{9}\times\overline{1-\frac{1}{6}z} = {\cdot}43484444$.

Then $\frac{2}{5}P\times\overline{A-B} = 1884{\cdot}955592 \times {\cdot}97797 = 1843{\cdot}4298 =$ the ſurface nearly.

EXAMPLE II.

Taking here the 2d example to the laſt rule, we have $P=prb=2513{\cdot}27412287$, $A=\sqrt{1+\frac{1}{3}z} = 1{\cdot}0583005$, and $B=\frac{4}{9}\times\overline{1+\frac{1}{6}z} = {\cdot}47\frac{1}{9}$.

Then $\frac{2}{5}P\times\overline{A-B} = 2513{\cdot}27412287\times 1{\cdot}0569409 = 2656{\cdot}3821 =$ the ſurface nearly.

RULE VII.

For both ſpheroids.

This rule, alſo, will be the ſame with the 7th to the laſt problem, uſing z and b inſtead of q and f.

That is, $\frac{27}{7} P \times \overline{A-B-C} = \frac{27}{7} prb \times \sqrt{\overline{1 \pm \frac{1}{3}z - \frac{4}{9}} \times \overline{1 \pm \frac{1}{6}z - \frac{7}{27}} \times \overline{1 \pm \frac{1}{6}z - \frac{1}{40}zz}}$ = the ſurface very nearly.

EXAMPLE I.

Taking, ſtill, the ſame example of the oblong ſpheroid, we ſhall have, as before, $P = 1884{\cdot}955592$, $A = {\cdot}97816157$, and $B = {\cdot}43484444$; alſo $C = \frac{8}{27} \times \overline{1 - \frac{1}{6}z - \frac{1}{40}zz} = {\cdot}28977188$.

Then $\frac{27}{7} P \times \overline{A-B-C} = 1884{\cdot}955592 \times {\cdot}97796 = 1843{\cdot}411$ = the ſurface very nearly.

EXAMPLE II.

Let there be taken, again, the oblate fruſtum, in which, as before, $P = 2513{\cdot}27412287$, $A = 1{\cdot}0583005$, and $B = {\cdot}47\frac{1}{9}$; alſo $C = \frac{8}{27} \times \overline{1 + \frac{1}{6}z - \frac{1}{40}zz} = {\cdot}31311407$.

Then $\frac{27}{7} P \times \overline{A-B-C} = 2513{\cdot}27412287 \times 1{\cdot}05714 = 2656{\cdot}8825$ = the ſurface very nearly.

SCHOLIUM.

It is evident that the double of the fruſtum will give the middle zone; and that the fruſtum being added to or taken from half the ſpheroid, will give the greater or leſs ſegment.

PROBLEM XI.

* *To find the ſolidity of a ſpheroid.*

RULE I.

Multiply continually together the fixed axe, the ſquare of the revolving axe, and the number ·52359877 &c. (= $\frac{1}{6}$ of 3·14159 &c.), and the laſt product will be the content required.

That

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* DEMONSTRATION.

Put $f = BI$ the fixed ſemi-axe, [fig. to prop. 1 ſect. 2] $r = IM$ the revolving ſemi-axe of the ſpheroid, $a = SI$ any ſemi-diameter of the ſection NBM, $b = IK$ its ſemi-conjugate, $y = AE$ an ordinate to the diameter SI, or a ſemi-axe of the elliptic ſection AFC parallel to KL, and $z = EF$ its other ſemi-axe, alſo $x = EI$, $s =$ the ſine of the angle SEA or of the angle SIK to the radius 1, and $p = 3{\cdot}14159$ &c.

Then, by the property of the ellipſe KSL, $aa : bb :: aa - xx : bb \times \frac{aa - xx}{aa} = yy$; and, by prop. 1 ſect. 2, $b : r :: y : \frac{ry}{b} = z$: but the fluxion of the ſolid $KACL$ is $psyz\dot{x} =$ (by writing for z its value $\frac{ry}{b}$) $\frac{psryy\dot{x}}{b} =$ (by ſubſtituting for yy its value $bb \times \frac{aa - xx}{aa}$) $pbsr\dot{x} \times \frac{aa - xx}{aa} =$ (by putting for abs its value rf) $pfrr\dot{x} \times \frac{aa - xx}{aaa}$; and hence the fluent or $pfrrx \times \frac{aa - \frac{1}{3}xx}{aaa}$ will be the value of the fruſtum $KACL$; which, when EI or x becomes IS or a, gives $\frac{2}{3}pfrr$ for the value of the ſemi-ſpheroid KSL; or the whole ſpheroid $= \frac{1}{6}pFRR$, putting F and R for the whole fixed and revolving axes. *Q.E.D.*

COROLLARY I.

From the foregoing demonſtration it appears that the value of the general fruſtum $KACL$ is expreſſed by $pfrrx \times \frac{aa - \frac{1}{3}xx}{aaa}$.

And

That is, $\frac{1}{6}pttc$ = the oblate, and $\frac{1}{6}ptcc$ = the oblong ſpheroid, where $p = 3{\cdot}14159$ &c. t = the tranſverſe, and c = the conjugate axe of the generating ellipſe.

RULE

And if for fr be ſubſtituted its value abs, it will alſo be expreſſed by $pbrsx \times \frac{aa - \frac{1}{3}xx}{aa}$.

Alſo, if for aa be put its value $\frac{bbxx}{bb - yy}$, the laſt expreſſion will become $prsx \times \frac{2bb + yy}{3b}$ or $psx \times \frac{2br + \frac{ryy}{b}}{3}$; which, by writing z inſtead of its value $\frac{ry}{b}$, gives $psx \times \frac{2br + yz}{3}$ for the value of the fruſtum, viz. The ſum of the area of the leſs end and twice that of the greater, drawn into one-third of the altitude or diſtance of the ends.—And out of this laſt expreſſion may be expunged any one of the four quantities b, r, y, z, by means of the proportion $b : r :: y : z$.

When the ends of the fruſtum are perpendicular to the fixed axe, then a is $= f$, and the value of the fruſtum becomes $prrxx \times \frac{ff - \frac{1}{3}xx}{ff}$ for the value of the fruſtum whoſe ends are perpendicular to the fixed axe, its altitude being x.

And when the ends of the fruſtum are parallel to the fixed axe, a is $= r$, and the expreſſion for ſuch a fruſtum becomes $pfx \times \frac{rr - \frac{1}{3}xx}{r}$.

COROLLARY II.

If to or from $\frac{2}{3}pfrr$, the value of the ſemi-ſpheroid, be added or ſubtracted $pfrrx \times \frac{aa - \frac{1}{3}xx}{aaa}$, the value of the general fruſtum $KACL$, there will reſult $pfrrhh \times \frac{a - \frac{1}{3}h}{aaa}$ for the value of a general ſegment, either greater or leſs than the ſemi-ſpheroid, whoſe height, taken upon the diameter paſſing through its vertex and center of its baſe, is $h = a \pm x$.

When

RULE II.

Multiply the area of the generating ellipſe by $\frac{2}{3}$ of the revolving axe, and the product will be the content of the ſpheroid.

That

When a coincides with f, the above expreſſion becomes $prrbb \times \frac{f - \frac{1}{3}b}{ff}$ for the value of a ſegment whoſe baſe is perpendicular to the fixed axe.—And here if we put R for the radius of the ſegment's baſe, and for rr its value $\frac{RRff}{2fb - bb}$, the expreſſion for the ſaid ſegment will become $\frac{1}{3}pRRb \times \frac{3f - b}{2f - b}$.

And when a coincides with r, the general expreſſion will become $pfbb \times \frac{r - \frac{1}{3}b}{r}$ for the value of the ſegment whoſe baſe is parallel to the fixed axe.—And if we put F, R, for the two ſemi-axes, of the elliptic baſe of this ſegment, reſpectively correſponding or parallel to f, r, the ſemi-axes of the generating ellipſe, when parallel to the baſe of the ſegment, and for $\frac{f}{r}$ and r ſubſtitute their values $\frac{F}{R}$ and $\frac{RR + bb}{2b}$, the ſaid fruſtum will be expreſſed by $\frac{1}{3}pFb \times \frac{3RR + bb}{2R}$, in which the dimenſions of itſelf only are concerned.

COROLLARY III.

A ſemi-ſpheroid is equal to $\frac{2}{3}$ of a cylinder or to double a cone of the ſame baſe and height, or they are in proportion as the numbers 3, 2, 1. For the cylinder is $= 4nfrr = \frac{12}{3}nfrr$, the ſemi-ſpheroid $= \frac{8}{3}nfrr$, and the cone $= \frac{4}{3}nfrr$.

COROLLARY IV.

When $f = r$, the ſpheroid becomes a ſphere, and the expreſſion $\frac{8}{3}fnrr$ for the ſemi-ſpheroid becomes $\frac{8}{3}nr^3$ for the ſemi-ſphere, as in prob. 24 ſect. 1.—And in like manner, f and r being ſuppoſed equal to each other in the values of the fruſtums and ſegments

of

That is, $\frac{2}{3}tA$ = the oblate, and $\frac{2}{3}cA$ = the oblong ſpheroid ; where A is the area of the ellipſe.—And this rule is evidently taken from the former.

of a ſpheroid, in the preceding corollaries, will give the values of the like parts of a ſphere.

COROLLARY V.

All ſpheres and ſpheroids are to each other as the fixed axes drawn into the ſquares of the revolving axes.

COROLLARY VI.

Any ſpheroids, and ſpheres, of the ſame revolving axe, as alſo their like or correſponding parts cut off by planes perpendicular to the ſaid common axe, are to one another as their other or fixed axes. This follows from the foregoing corollaries.

COROLLARY VII.

But if their fixed axes be equal, and their revolving axes unequal, the ſpheroids and ſpheres, with their like parts terminated by planes perpendicular to the common fixed axe, will be to each other as the ſquares of their revolving axes.

COROLLARY VIII.

An oblate is to an oblong ſpheroid generated from the ſame ellipſe, as the longer axe of the ellipſe is to the ſhorter. For, if T be the tranſverſe axe and C the conjugate, the oblate ſpheroid will be $= \frac{2}{3}nT^2C$, and the oblong $= \frac{2}{3}nC^2T$; and theſe quantities are in the ratio of T to C.

COROLLARY IX.

And if about the two axes of an ellipſe be generated two ſpheres and two ſpheroids, the four ſolids will be continual proportionals whoſe common ratio will be that of the two axes of the ellipſe ; that is, as the ſphere upon the greater axe, or greater ſphere, is to the oblate ſpheroid, ſo is the oblate ſpheroid to the oblong ſpheroid, ſo is the oblong ſpheroid to the leſs ſphere, and ſo is the tranſverſe axe to the conjugate. For theſe four bodies will be as T^3, T^2C, TC^2, C^3, where each term is to the conſequent one as T to C.

EXAMPLE.

Required the content of an oblate and of an oblong ſpheroid, the axes being 50 and 30.

Firſt, $50 \times 30 \times \cdot 78539816 = 1178 \cdot 09724$ = the area of the ellipſe.

Then $1178 \cdot 09724 \times \frac{2}{3} \times 30 = 23561 \cdot 9448$ = the oblong ſpheroid.

And $1178 \cdot 09724 \times \frac{2}{3} \times 50 = 39269 \cdot 908$ = the oblate one.

PROBLEM XII.

To find the content of the fruſtum of a ſpheroid, its ends being perpendicular to one of the axes, and one of them paſſing through the center.

RULE I.

To the area of the leſs end add twice that of the greater, multiply the ſum by the altitude of the fruſtum, and $\frac{1}{3}$ of the product will be the content.—By corollary 1 to the laſt problem.

That is, $\overline{2D^2 + d^2} \times \frac{1}{3}an$ = the fruſtum whoſe ends are perpendicular to the fixed axe, where D is the diameter of the greater end, d is that of the leſs, a the altitude, and $n = \cdot 785398$ &c.

And $\overline{2TC + tc} \times \frac{1}{3}an$ = the fruſtum whoſe ends are parallel to the fixed axe, where T and C are the tranſverſe and conjugate axes of the greater end, and t and c thoſe of the leſs end.

Note. It is evident that the double of the fruſtum will give the content of the zone, or ſpheroidal caſk.

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EXAMPLE I.

There is a caſk in the form of the middle fruſtum or zone of an oblong ſpheroid, the bung diameter is 30, the head diameter 18, and the length of the caſk is 40 inches; what is the content in ale and wine gallons.

Here $D = 30$, $d = 18$, and $a = 40$.

Wherefore $\overline{2D^2 + d^2} \times \frac{1}{3}an = \overline{2 \times 30^2 + 18^2} \times \cdot2618 \times 40 = 2124 \times 10\cdot472 = 22242\cdot528 =$ the content in inches.

Then, ſince the gallon ale meaſure contains 282 cubic inches, and the wine gallon 231,

We have $\frac{22242\cdot528}{282} = 78\cdot874$ ale gallons.

And $\frac{22242\cdot528}{231} = 96\cdot288$ wine gallons.

EXAMPLE II.

If a veſſel, in the form of the middle fruſtum of an oblate ſpheroid, have the diameter of each end 40, in the middle 50, and its length 18 inches; what is its content in ale and wine gallons.

Here $\overline{2D^2 + d^2} \times \frac{1}{3}an = \overline{2 \times 50^2 + 40^2} \times \cdot2618 \times 18 = 118800 \times \cdot2618 = 31101\cdot84$ cubic inches.

Then $\frac{31101\cdot84}{282} = 110\cdot29$ ale gallons,

And $\frac{31101\cdot84}{231} = 134\cdot64$ wine gallons.

EXAMPLE III.

In the fruſtum of an oblong ſpheroid, the greater end is the generating ellipſe whoſe axes are 50 and 30, the axes of its leſs end 40 and 24, and its height nine inches; required the content in wincheſter buſhels.

Here $\overline{2TC+tc}\times\frac{1}{3}an = \overline{2\times 50\times 30+40\times 24}\times$ $\cdot 2618\times 9 = 9330{\cdot}552$ cubic inches.

But $268{\cdot}8$ inches = a gallon, or $2150{\cdot}4$ = a corn buſhel.

And therefore $\frac{9330\cdot 552}{2150\cdot 4} = 4{\cdot}339$ buſhels.

EXAMPLE IV.

In the fruſtum of an oblate ſpheroid the greater end is the generating ellipſe whoſe axes are 50 and 30, and the height is 20 inches; required the ſolidity.

Here, by the nature of the ellipſe, as $50 : 30 ::$ $\sqrt{\overline{25+20}\times\overline{25-20}} = \sqrt{45\times 5} = 15 : \frac{15\times 30}{50} = 9$ = the ſemi-conjugate axe of the leſs end.

And, by prop. 1 ſect. 2, as $30 : 50 :: 9 : 15$ = the ſemi-tranſverſe axe of the leſs end.

Then $\overline{2TC+tc}\times\frac{1}{3}na = \overline{2\times 50\times 30+30\times 18}\times$ $\cdot 2618\times 20 = 118\times 30\times 20\times\cdot 2618 = 70800\times\cdot 2618$ $= 18535{\cdot}44$ cubic inches.

RULE II.

From 3 times the ſquare of the ſemi-axe perpendicular to the ends of the fruſtum ſubtract the ſquare of the height of the fruſtum, then multiply the difference by $\frac{1}{3}$ of the height, and the product by $3{\cdot}14159$ &c. and call the laſt product *P*.—Then

1. *Iſ*

1. *If the ends be parallel to the fixed axe,* As the revolving axe is to the fixed axe, ſo will P be to the content of the fruſtum.

2. *When the ends are perpendicular to the fixed axe,* As the ſquare of the fixed axe is to the ſquare of the revolving axe, ſo is P to the content of the fruſtum.

That is, $\frac{3ff-hh}{3ff} \times phrr$ will be the fruſtum whoſe ends are perpendicular. And $\frac{3rr-hh}{3r} \times phf$ $=$ that whoſe ends are parallel to the fixed axe; f being the ſemi-fixed and r the ſemi-revolving axe, h the height, and $p = 3{\cdot}14159$ &c.

EXAMPLE I.

If the axes of an oblong ſpheroid be 50 and 30, required the content of a fruſtum whoſe ends are perpendicular to the fixed axe, and one of them paſſing through the center, the height of the fruſtum being 20 inches.

Here $\frac{3ff-hh}{3ff} \times phrr = \frac{3 \times 25^2 - 20^2}{3 \times 25^2} \times 3{\cdot}14159$ &c. $\times 20 \times 15^2 = 1475 \times 3{\cdot}14159$ &c. $\times \frac{12}{5} = 11121{\cdot}23799$ $=$ the content required.

EXAMPLE II.

If from the ſame ſpheroid be cut a fruſtum whoſe height is 9, the ends being parallel to the fixed axe, and one of them paſſing through the center, what will be its content.

Here $\frac{3rr-hh}{3r} \times phf = \frac{3 \times 15^2 - 9^2}{3 \times 15} \times 9 \times 25 \times 3{\cdot}14159$ &c. $= 594 \times 5 \times 3{\cdot}14159$ &c. $= 9330{\cdot}53018 =$ the content required.

EXAMPLE III.

If from an oblate ſpheroid whoſe axes are 50 and 30, a fruſtum whoſe height is 9 and whoſe ends are perpendicular to the fixed axe be cut, what will be its ſolidity.

Here $\frac{3ff-hh}{3ff}\times phrr=\frac{3\times15^2-9^2}{3\times15^2}\times9\times25^2\times$ $3{\cdot}14159=594\times\frac{25}{3}\times3{\cdot}14159$ &c. $=15550{\cdot}88363$ = the content required.

EXAMPLE IV.

In the ſame ſpheroid it is required to find the content of a fruſtum whoſe height is 20, the ends being parallel to the fixed axe.

Here $\frac{3rr-hh}{3r}\times phf=\frac{3\times25^2-20^2}{3\times25}\times20\times15\times$ $3{\cdot}14159$ &c. $=1475\times4\times3{\cdot}14159$ &c. $=18535{\cdot}39665$ = the content required.

PROBLEM XIII.

To find the content of a ſpheroidal caſk, not full, ſtanding upon its end, the axe being perpendicular to the horizon. That is, of the fruſtum of an oblong ſpheroid, the ends being perpendicular to the axe, but neither of them paſſing through the center.

RULE.

Multiply the difference of the ſquares of the bung and head diameters by 4 times the ſquare of the difference between the height of the liquor and half the length of the caſk, and divide the product by the ſquare of the length of the caſk; ſubtract the quotient from 3 times the ſquare of the bung diameter, and multiply the remainder by the aforeſaid difference between the height of the liquor and $\frac{1}{2}$ the length of the caſk; then the product multiplied by ·261799 &c. (= $\frac{1}{3}$ of ·78539 &c.) will produce the quantity by which the caſk is more or leſs than half full; and which, therefore, being added to or taken from half the content of the caſk, will give the content of the part filled.

That is, $\overline{2b^2 + h^2} \times \frac{1}{3} l \pm \overline{3b^2 - \frac{4dd}{ll} \times \overline{b^2 - h^2}} \times d \times \frac{1}{3} n =$ the content of the part filled, called the ullage, uſing the upper or under ſigns according as the caſk is more or leſs than half full; where b and h = the bung and head diameters, l = the length of the caſk, n = ·785398 &c. and $d = \frac{1}{2} l \backsim w$, w being the wet part of l or the height of the liquor in the caſk.*

Ex-

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* DEMONSTRATION.

For, if D be the diameter at the ſurface of the liquor, by the property of the ellipſe $b^2 - h^2 : \frac{1}{4} l^2 :: b^2 - D^2 : d^2$, and hence $D^2 = b^2 - \frac{4dd}{ll} \times \overline{b^2 - h^2}$.

But, by the laſt problem, $\overline{2b^2 + h^2} \times \frac{1}{6} ln =$ half the content of the caſk, and $3b^2 - \frac{4dd}{ll} \times \overline{b^2 - h^2} \times \frac{1}{3} dn =$ the fruſtum by which the part filled is more or leſs than half the caſk.

Conſequently $\overline{2b^2 + h^2} \times \frac{1}{3} l \pm 3b^2 - \frac{4dd}{ll} \times \overline{b^2 - h^2} \times d \times \frac{1}{3} n =$ the part filled.

EXAMPLE I.

If a ſpheroidal caſk whoſe head and bung diameters are 18 and 30 inches, and length 40 inches, be filled to the height of 30 inches; how many ale and wine gallons are in it.

Here $b = 30$, $h = 18$, $l = 40$, $w = 30$, and $d = \frac{1}{2}l \backsim w = 30 - 20 = 10$.

Then $\overline{3b^2 - \frac{4dd}{ll} \times \overline{b^2 - h^2}} \times d = \overline{2700 - \frac{400}{1600} \times 48 \times 12} \times 10 = \overline{2700 - 144} \times 10 = 25560$.

And $\overline{2b^2 + h^2} \times \frac{1}{2}l = \overline{1800 + 18 \times 18} \times 20 = 42480$.

Conſequently $\overline{42480 + 25560} \times \frac{1}{3}n = 68040 \times {}^{\cdot}2617993878 = 17812{\cdot}8303458$ = the content in inches; and being divided by 282 and 231, gives 63·166065 ale, and 77·111819 wine gallons.

EXAMPLE II.

If the height of the liquor in the ſame caſk be only 10 inches, required the ullage.

Here $d = \frac{1}{2}l \backsim w = 20 - 10 = 10$ the ſame as before, and therefore the part which in the laſt example was added muſt here be ſubtracted; ſo that $\overline{42480 - 25560} \times \frac{1}{3}n = 16920 \times {}^{\cdot}2617993878 = 4429{\cdot}6456415$ the content in inches = 15·707963 ale, and 19·175955 wine gallons.

PRO-

PROBLEM XIV.

To find the ſolidity of the ſegment of a ſpheroid whoſe baſe is perpendicular to one of the axes.

RULE I.

From 3 times the ſemi-axe perpendicular to the baſe of the ſegment, take the height of the ſegment, multiply the remainder into $\frac{1}{3}$ of the ſquare of the height, then multiply the product by 3·14159 &c. and call the laſt product Q.

Divide the axe which is parallel to the baſe by the other axe, and call the quotient q.

Then for the ſegment whoſe baſe is perpendicular to the fixed axe, multiply Q into the ſquare of q; and for the other ſegment, multiply Q into q.

That is, $Qqq = \frac{3f-h}{3ff} \times prrhh$ = the ſegment whoſe baſe is perpendicular to the fixed axe;—And $Qq = \frac{3r-h}{3r} \times pfhh$ = the other ſegment whoſe baſe is parallel to the fixed axe; where h is the height of the ſegment, and the other ſymbols as in the laſt problem.

EXAMPLE I.

If from an oblong ſpheroid, whoſe axes are 50 and 30, be cut a ſegment whoſe baſe is perpendicular to the fixed axe, its height being 5; required the content of it.

Here $\frac{3f-h}{3ff} \times prrhh = \frac{3\times25-5}{3\times25^2} \times 15^2 \times 5^2 \times 3{\cdot}14159$ &c. $= 70 \times 3 \times 3{\cdot}14159$ &c $= 659{\cdot}73445$ = the content required.

EXAMPLE II.

If from the ſame ſpheroid be cut a ſegment whoſe height is 6, what will its content be, ſuppoſing its baſe to be parallel to the fixed axe.

Here $\frac{3r-h}{3r}\times p\text{ſ}hh = \frac{3\times15-6}{3\times15}\times25\times6^2\times3\cdot14159$ &c. $= 39\times20\times3\cdot14159$ &c. $= 2450\cdot44226 =$ the content required.

EXAMPLE III.

Required the ſolidity of the ſegment of an oblate ſpheroid whoſe axes are 50 and 30, the height of the ſegment being 6, and its baſe perpendicular to the fixed axe.

Here $\frac{3\text{ſ}-h}{3\text{ſſ}}\times prrhh = \frac{3\times15-6}{3\times15\times15}\times25\times25\times6\times6\times$ $3\cdot14159$ &c. $= 1300\times3\cdot14159$ &c. $= 4084\cdot07044$ = the content required.

EXAMPLE IV.

To find the content of a ſegment of the ſame ſpheroid, its baſe being parallel to the fixed axe, and its height 5.

Here $\frac{3r-h}{3r}\times p\text{ſ}hh = \frac{3\times25-5}{3\times25}\times15\times5\times5\times3\cdot14159$ &c. $= 70\times5\times3\cdot14159$ &c. $= 1099\cdot55742 =$ the content required.

RULE II.

For the ſegment whoſe baſe is parallel to the fixed axe, having given its height and the diameters of its end.

To the ſquare of the height add 3 times the ſquare of the leſs or greater ſemi-axe of the baſe according as it is the ſegment of an oblong or oblate ſpheroid, divide the ſum by the ſame ſemi-axe, and multiply the quotient by the other ſemi-axe, the product by the height, and this product by ($\frac{1}{6}$ of 3·14159 &c.) ·52359 &c. will give the content of the ſegment.

That is, $\frac{3A^2+b^2}{6A} \times pBh =$ the content of the ſegment, where h is its height, A the leſs or greater ſemi-diameter of the baſe according as the ſegment is that of an oblong or oblate ſpheroid, B is the other ſemi-diameter, and $p = 3\cdot14159$ &c.

EXAMPLE I.

It is required to find the content of the ſegment of an oblong ſpheroid, whoſe baſe is parallel to the fixed axe, its height being 6, and the axes of its elliptic baſe 40 and 24.

Here $\frac{3AA+bb}{6A} \times pBh = \frac{3\times12^2+6^2}{6\times12} \times 20\times6\times3\cdot14159$ &c. $= 13\times60\times3\cdot14159$ &c. $= 2450\cdot44226 =$ the content required.

EXAMPLE II.

Required the content of the ſegment of an oblate ſpheroid, whoſe baſe is parallel to the fixed axe, its height being 5, and the diameters of its baſe 18 and 30.

Here $\frac{3A^2+b^2}{6A} \times pBh = \frac{3\times15^2+5^2}{6\times15} \times 9\times5\times3\cdot14159$ &c. $= 70\times5\times3\cdot14159$ &c. $= 1099\cdot55742 =$ the content required.

RULE III.

For a ſegment whoſe baſe is perpendicular to the fixed axe, having given the height, the diameter of its baſe, and a diameter in the middle between its baſe and vertex.

To the ſquare of the diameter of the baſe add 4 times the ſquare of the diameter in the middle, or the ſquare of twice this diameter; multiply the ſum into the height, and the product multiplied by ($\frac{1}{8}$ of ·78539 &c.) ·13089969 &c. will give the content.

That is, $\overline{D^2+4d^2}\times\frac{1}{8}nh =$ the content of the ſegment; D being the diameter of the baſe, $d =$ the diameter in the middle, $h =$ the height, and $n =$ ·785398 &c.*

Ex-

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* DEMONSTRATION.

By rule 1 the ſegment is $= 4nrrhh\times\frac{3f-h}{3ff}$. But, by the nature of the ellipſe, D is $= \frac{2r\sqrt{2fh-hh}}{f}$, and $d = \frac{2r\sqrt{fh-\frac{1}{4}hh}}{f}$; hence $f = h\times\frac{4dd-DD}{8dd-4DD}$, and $\frac{r}{f} = \frac{\sqrt{4dd-2DD}}{2h}$; which values being ſubſtituted in the above value of the ſegment, gives $\frac{1}{8}nh\times\overline{DD+4dd}$ for the value of the ſegment in terms of D, d, and h.

SCHOLIUM.

This theorem I have inveſtigated in order to expreſs the value of this ſegment independent of the axes of the ſpheroid. For this purpoſe I was under a neceſſity of introducing another dimenſion of the ſegment, becauſe it is not determinable from its baſe and height alone as the other ſegment was.

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EXAMPLE I.

What is the content of the ſegment of an oblong ſpheroid, whoſe baſe is perpendicular to the fixed axe, its height being 5, the diameter of its baſe 18, and its middle diameter $3\sqrt{19}$.

Here $\overline{DD+4dd} \times \frac{1}{6}nb = \overline{18^2 + \overline{6\sqrt{19}}|^2} \times \frac{5}{6} \times$ ·785398 &c. = $\overline{3^2+19} \times 30 \times$ ·785398 &c. = 210 × 3·14159 &c. = 659·73445 = the content required.

EXAMPLE. II.

Required the content of the ſpheroidal ſegment whoſe height is 6, its baſe diameter 40, and diameter in the middle 30, the baſe being perpendicular to the fixed axe of the ſpheroid.

Here $\overline{DD+4dd} \times \frac{1}{6}bp = \overline{40^2+60^2} \times \frac{6}{6} \times$ ·785398 &c. = $\overline{2^2+3^2} \times 20^2 \times$ ·785398 = 1300 × 3·14159 &c. = 4084·07044 = the content required.

PROBLEM XV.

To find the content of the ſecond ſegment of a ſpheroid.

RULE.

As a ſphere is to the ſpheroid, ſo is any part of the ſphere to the like part of the ſpheroid.*

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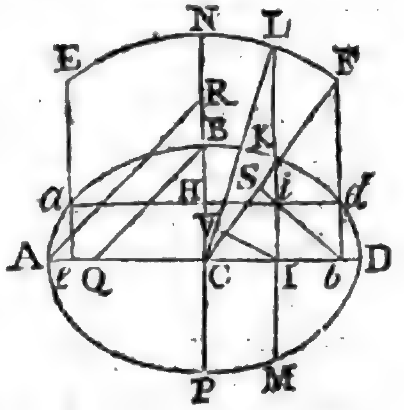
* DEMONSTRATION.

For the like parts of any quantities are as the wholes. And that the ſecond ſegments, *FGH*, *fgh*, are like parts of the ſphere and ſpheroid is evident from the nature of the figures.

PROBLEM XVI.

To find the ſurface of an elliptic ſpindle, or of any fruſtum or ſegment of it.

GENERAL RULE.



From the ſpheroidal ſurface generated by any arc *BK* of the ellipſe, take the product of the ſame arc *BK* and the circumference of the circle whoſe diameter is equal to twice *CH* the diſtance of

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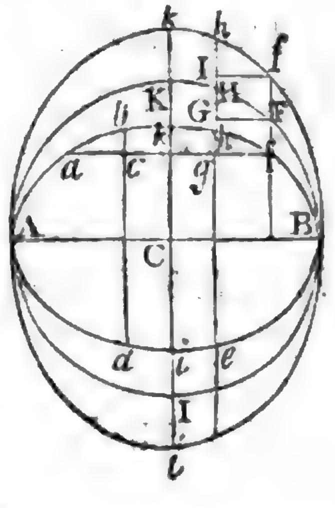
COROLLARY I.



And if the ſphere *AKBI* and ſpheroid *AkBi* have one axe *AB* common; then the whole ſolids and the ſecond ſegments *FGH*, *fgh*, will be to each other as the ſquares of the other axes *KI*, *ki*, if the common axe *AB* be the fixed one; or they will be to each other as the other axes *KI*, *ki*, ſimply, if the common axe be the revolving one. For, if *A* be the axe of the ſphere, *F* and *R* the fixed and revolving axes of the ſpheroid, the bodies will be to each other as A^3 to FR^2; hence if $A = F$, they will be as A^2 to R^2; but if $A = R$, they will be as *A* to *F*.

COROLLARY II.

Hence may be found the true quantity of liquor in a ſpheroidal caſk, not full, whoſe axe is parallel to the horizon.

For if from the ſegment *akf* be taken the double of the ſecond ſegment *fgh*, there will remain the part *hhgc*; which taken from the whole caſk *bhed* will leave the part *cdeg*.

And after the ſame manner may be found the quantity of liquor in a ſpheroidal caſk partly filled and ſtanding a-tilt with its axe inclined to the horizon.

of the centers of the ſpheroid and ſpindle, and the remainder will be the ſurface of the ſpindle generated by the arc BK about ad parallel to AD the fixed axe of the ſpheroid.*

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* DEMONSTRATION.

For the fluxion $\dot{s}$ of the ſpindular ſurface is $= \dot{z}$ the fluxion of the arc BK drawn into $iK \times 2p = 2p\dot{z} \times \overline{KI - Ii}$, and the fluent is equal to the ſpheroidal ſurface $BKMP - 2pz \times Ii$ or CH.

COROLLARY I.

The ſpindular ſurface generated by BK is equal the ſpheroidal ſurface $BKMP$ — the ſurface of a ſphere whoſe axe is a mean proportional between BK and $2CH$. For $2pz \times CH =$ that ſpheric ſurface.—Or $2pz \times CH =$ a circle whoſe radius is $\sqrt{2CH \times BK}$.

COROLLARY II.

When CH is equal nothing, the ſpindle becomes barely a ſpheroid. And when H falls below C, the ſurface of the ſphere muſt be added.—What hath been hitherto done agrees to the ſurface generated by an arc about a line parallel to either axe of the ellipſe.

COROLLARY III.

From B the extremity of the leſs axe of the ellipſe, apply $BQ = AC$ the ſemi-tranſverſe, and parallel thereto draw AR meeting CB, produced, in R; with the center C and the radius AR deſcribe the arc EF meeting the parallels aE, CB, IK, bd in E, L, N, F: Draw CL and CF, and perpendicular to them IV and bS.—Then the circle whoſe radius is equal to a mean proportional between the ſum and difference of two lines whereof the one is a mean proportional between BC and $\overline{IV + LN}$, and the other a mean proportional between BK and $2CH$ will be equal to the ſpindular ſurface generated by BK about $ab \parallel$ the tranſverſe axe. For the ſpheroidal ſurface is equal to a circle whoſe radius is a mean proportional between BC and $\overline{IV + LN}$ by rule 4 to prob. 9, and $2p \times BK \times CH$ is equal to a circle whoſe radius is a mean proportional between BK and $2CH$; but the ſpindular ſurface is equal to the difference of thoſe two quantities, and the difference of two circles is equal to a circle whoſe radius is a mean proportional between the ſum and difference of the radiuſes of the two circles, therefore, &c.—And, in the ſame manner, the

EXAMPLE I.

Given the axes of an ellipſe 50 and 40, to find the ſurface of the ſpindle generated from an arc of that ellipſe, the length of the ſpindle being 30.

Here $AD = 50$, $BP = 40$, $ad = be = 30$, $Ae = AC - Ce = 25 - 15 = 10$, and $eD = DC + Cb = 25 + 15 = 40$. Then as $AC : CB :: \sqrt{Ae \times eD} : ea = CH = \frac{CB\sqrt{Ae \times eD}}{AC} = \frac{20\sqrt{10 \times 40}}{25} = \frac{4 \times 20}{5} = 16$ = the diſtance of the centers of the ellipſe and ſpindle.

By example 1 to problem 10, the ſpheroidal ſurface generated by the arc *ad* is 3686·8197024, and by prob. 4, the length of the elliptic arc *ad* is 30·852.

Wherefore $3686{\cdot}8197 - 30{\cdot}852 \times 3{\cdot}1416 \times 32 = 3686{\cdot}8197 - 3101{\cdot}5872 = 585{\cdot}2325$ = the ſurface of the ſpindle.

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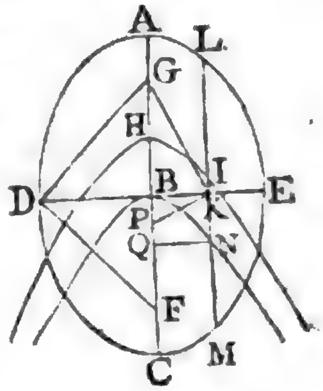
the ſurface of half the ſpindle generated by the arc Bd is equal to a circle whoſe radius is a mean proportional between the ſum and difference of two lines, the one of which is a mean proportional between BC and $\overline{bS + NF}$, and the other a mean proportional between Bd and $2CH$.

COROLLARY IV.

In like manner, for the ſurface of the ſpindle whoſe axe is parallel to the leſs axe of the ellipſe: having conſtructed the annexed figure as in page 259, the ſpindular ſurface generated by the arc CM about NQ will be equal to a circle whoſe radius is a mean proportional between the ſum and difference of two lines whereof the one is a mean proportional between AC and HI, and the other a mean proportional between GM and $2BQ$.—This, like the laſt corollary, will be evident by comparing what is done above with what is done in page 259.



EXAMPLE II.

Required the ſurface of the fruſtum of a ſpindle generated from an arc of the ſame ellipſe as in the laſt example, the height of the fruſtum being 15, and the central diſtance 10.

Here the arc *EQ* generating the fruſtum *EQIC* is equal to the arc in the laſt; and conſequently that arc is = 15·426, and the ſpheroidal fruſtum *EQFD* = 1843·40985.

Then 1843·4098 − 15·426 × 3·1416 × 20 = 1843·4098 − 969·246 = 874·163 = the ſurface of the fruſtum *EQIC* required.

PROBLEM XVII.

To find the ſolidity of an elliptic ſpindle.

RULE I.

1. Divide the ſquare of the perpendicular axe *DE* by 3 times the ſquare of the parallel axe *AB*, and multiply the quotient by the cube of *FG* the axe or length of the ſpindle, and call the product *P*.

2. Find the area of the elliptic ſegment *FDG* from which the ſpindle is generated, multiply this area by 4 times *CH* the central diſtance, and call the product *Q*.

3. Multiply 1·57079 &c. by the difference between *P* and *Q*, and the product will be the content of the ſpindle *FDGN*.

That is, $2n \times \overline{\frac{aa}{3bb}l^3 - 4cS} =$ the ſpindle $FDGN$, when $n = {\cdot}78539$ &c. $a = DE$, $b = AB$, $l = FG$, $c = CH$, and $S =$ the ſegment FDG.*

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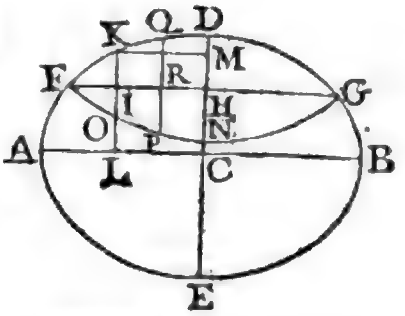
* DEMONSTRATION.



Let $FDGN$ be a ſpindle generated by the arc FDG of the ellipſe $ADBE$. Put $a = FH =$ half the axe of the ſpindle; $b = CH = LI =$ the central diſtance, or diſtance of the centers of the ellipſe and ſpindle; $c = AC =$ the one ſemi-axe of the ellipſe, and $d = CD =$ the other ſemi-axe; $x = HI = CL$; and $y = IK = HM$.

By the property of the ellipſe, $c : d :: \sqrt{cc - xx} : \frac{d}{c}\sqrt{cc - xx} = LK$, hence $IK = KL - LI = \frac{d\sqrt{cc - xx}}{c} - b = y$, and the fluxion of the ſolid $\dot{S} = pyy\dot{x} = p\dot{x} \times \overline{dd - \frac{ddxx}{cc} - \frac{2bd\sqrt{cc - xx}}{c} + bb} = p\dot{x} \times \overline{dd - bb - \frac{ddxx}{cc} - 2b \times \overline{\frac{d\sqrt{cc - xx}}{c} - b}} = pdd\dot{x} \times \frac{aa - xx}{cc} - 2pby\dot{x}$; and the fluent is $S = pddx \times \frac{3aa - xx}{3cc} - 2bp \times$ area $HIKD =$ the fruſtum $DKON$.—When x is $= a$, the above theorem will become $\frac{2pdd}{3cc} \times a^3 - 2bp \times$ area $FDH = DFN$ the half of the ſpindle.—And if from the ſemi-ſpindle be taken the fruſtum, there will remain $pddee \times \frac{3a - e}{3cc} - 2bp \times$ area $FKI =$ the ſegment FKO of the ſpindle, e being the height FI of the ſegment.

COROLLARY I.

If d be ſuppoſed $= c$, the ellipſe will become a circle, and accordingly the theorems above given will become the ſame with thoſe before found for the circular ſpindle.

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EXAMPLE.

The axes of an ellipſe being 50 and 30, required the ſolidity of a ſpindle generated from an arc

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COROLLARY II.

If H coincide with C, b will vaniſh, a will be $= c$, and the theorems will become the ſame with thoſe before found for the ſpheroid.

COROLLARY III.

Putting $D = DN$ the greateſt, and $d = KO$ the leaſt diameter of the fruſtum, $h = HI$ its height, $C = CH$ the central diſtance, $s =$ the elliptic ſemi-ſegment KDM, and $n = \cdot 78539$ &c. Then, in the foregoing theorems, $h = x$, $b = C$, $d = C + \frac{1}{2}D$, area $HIKD = s + \frac{1}{2}dh$, $c = b \times \frac{C+\frac{1}{2}D}{\sqrt{\frac{1}{4}DD + DC - \frac{1}{4}dd - dC}}$, and $a = b\sqrt{\frac{\frac{1}{4}DD + DC}{\frac{1}{4}DD + DC - \frac{1}{4}dd - dC}}$; which values being ſubſtituted in the theorems above, give $\frac{1}{3}nh \times \overline{2DD + dd - 8C \times \overline{-D+d} + \frac{3S}{h}}$ for the value of the fruſtum, or half a caſk in the form of the middle zone of an elliptic ſpindle; and $\frac{1}{3}nl \times \overline{2DD - 8C \times -D + \frac{3S}{l}}$ for half the ſpindle when $S =$ the area DFH, and $l = HF$.

COROLLARY IV.

But in real practice, ſuch as caſk gauging, none of theſe rules can be uſed, becauſe that we have not given either the axes of the ellipſe or the central diſtance; and to accommodate rules to that purpoſe we muſt introduce another dimenſion of a fruſtum beſides its length and greateſt and leaſt diameters; Thus, putting $m = PQ$ the diameter through R the middle of the length HI, and the other letters as before. Then, by the property of the ellipſe, $\overline{C+\frac{1}{2}D}|^2 - \overline{C+\frac{1}{2}d}|^2 : 4 :: \overline{C+\frac{1}{2}D}|^2 - \overline{C+\frac{1}{2}m}|^2 : 1$, hence $4 \times \overline{C+\frac{1}{2}m}|^2 - 3 \times \overline{C+\frac{1}{2}D}|^2 = \overline{C+\frac{1}{2}d}|^2$, and $C = \frac{1}{4} \times \frac{3D^2 + d^2 - 4m^2}{-3D - d + 4m}$. And the laſt two theorems become $\frac{1}{3}nh \times \overline{2D^2 + d^2 - 2 \times \frac{3D^2 + d^2 - 4m^2}{-3D - d + 4m} \times \overline{-D + d} + \frac{3s}{h}}$ for the fruſtum $DKON$, and $\frac{1}{3}nl \times \overline{2D^2 - 2 \times \frac{3D^2 - 4m^2}{-3D + 4m} \times -D + \frac{3S}{l}}$ for the ſemi-ſpindle DFN.

arc of it about its chord parallel to, and at the diſtance of, 9 from the tranſverſe axe.

Here $15-9=6=DH=$ the height of the generating ſegment; and, by the property of the ellipſe, $DC:CA::2\sqrt{DH\times HE}:\frac{2AC\sqrt{DH\times HE}}{CD}=\frac{50\sqrt{6\times24}}{15}$ $=\frac{10\times12}{3}=40=FG$ the baſe of the elliptic ſegment, or length of the ſpindle; alſo, by prob. 6, the area of the ſegment is 167·7345.

Then $\frac{3{\cdot}14159\text{ \&c.}}{2}\times\overline{\frac{30\times30}{3\times50\times50}\times40^3-4\times9\times167{\cdot}7345}$ $=18\times3{\cdot}14159\text{ \&c.}\times\overline{213\tfrac{1}{3}=167{\cdot}7345}=2578{\cdot}55=$ the content required.

RULE II.

Divide 3 times the generating ſegment by the length of the ſpindle, from the quotient ſubtract the greateſt diameter of the ſpindle, multiply the remainder by 4 times the central diſtance, and ſubtract the product from the ſquare of the greateſt diameter; then the difference multiplied by $\frac{1}{3}$ of the length of the ſpindle, and the product by 1·57079 &c. will give the content of the ſpindle.

That is, $\frac{2}{3}nl\times\overline{D^2-4c\times\overline{-D+\frac{3S}{l}}}=$ the ſpindle, where $D=DN$ the greateſt diameter, and the reſt of the ſymbols as in the laſt rule.—By corollary 3.

EXAMPLE.

Required the content of an elliptic ſpindle whoſe length is 40, greateſt diameter 12, and the central diſtance 9.

Here $\frac{12}{2} = 6 =$ the height of the generating ſegment, which is therefore the ſame as before, the area being 167·7345.

Then $3·14159\ \&c. \times \frac{40}{6} \times \overline{12^2 - 4 \times 9 \times \overline{-12 + \frac{3 \times 167·7345}{40}}}$ $= 3·14159\ \&c. \times 80 \times \overline{12 - 1·74024} = 2578·56 =$ the content required.

RULE III.

From three times the ſquare of the greateſt dia meter take 4 times the ſquare of the diameter in the middle between the greateſt diameter and the end. And from 4 times the ſaid middle diameter take 3 times the ſaid greateſt diameter. Divide the former difference by the latter, and $\frac{1}{4}$ of the quotient will be the central diſtance.

That is, $\frac{1}{4} \times \frac{3DD - 4mm}{4m - 3D} =$ the central diſtance, D being the greateſt diameter and m the middle diameter.

Then proceed as in the laſt rule.—This is proved in corollary 4.

EXAMPLE.

If, as in the example to the laſt rule, the greateſt diameter be 12, and the length 40, required the content ſuppoſing the diameter at $\frac{1}{4}$ of the length to be $6 \times \overline{\sqrt{21} - 3} = 9·49546$.

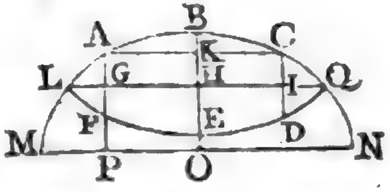
Here $\frac{1}{4} \times \frac{3DD - 4mm}{4m - 3D} = \frac{1}{4} \times \frac{3 \times 12^2 - 4 \times 6^2 \times \overline{\sqrt{21} - 3}|^2}{4 \times 6 \times \overline{\sqrt{21} - 3} - 3 \times 12}$ $= \frac{12}{4} \times \frac{3 - 21 + 6\sqrt{21} - 9}{2\sqrt{21} - 6 - 3} = 3 \times \frac{6\sqrt{21} - 27}{2\sqrt{21} - 9} = 3 \times 3 = 9 =$ the central diſtance, the ſame as in the laſt example; and therefore the content will be 2578·56, as before.

PROBLEM XVIII.

To find the ſolidity of the fruſtum of an elliptic ſpindle, or the content of a caſk in form of the middle zone of ſuch a ſpindle.

RULE I.

1. Divide the ſquare of the leſs axe of the ellipſe by the ſquare of the greater, multiply the quotient by $\frac{1}{3}$ of the length of the fruſtum or of half the length of the caſk, multiply the product by the difference between the ſquare of the ſaid half length of the caſk and 3 times the ſquare of half the length of the whole ſpindle, and call this product P.



That is, $P = ccl \times \frac{3LL - ll}{3tt}$, t being the tranſverſe MN, and c the conjugate axe $2BO$ of the ellipſe, $l = GH$ half the length of the caſk, and $L = LH$ half the length of the whole ſpindle.

2. Multiply OH the central diſtance by $ABHG$ the generating area, and call double the product Q.
That is, $Q = 2OH \times ABHG$.

3. Then the difference between P and Q multiplied by 3·14159 &c. will give the content of the fruſtum $ABEF$, or half the caſk $ACDF$.

That is, $\overline{P - Q} \times p = ABEF = \frac{1}{2}ACDF$.—By the demonſtration of the laſt problem.

NOTE.

It is evident that the ſum or difference of two fruſtums will give the ullage of a caſk ſtanding with its axe perpendicular to the horizon.

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EXAMPLE.

The axes of an ellipſe being 40 and $66\frac{2}{3}$; it is required to find the content of the fruſtum of a ſpindle generated by an arc of the ellipſe; the length of the fruſtum being 20, and the central diſtance 4.

Here, as $40 : 66\frac{2}{3} :: \sqrt{16 \times 24} : 13\frac{1}{3}\sqrt{6} = HL =$ half the length of the ſpindle.

And $ccl \times \frac{3LL-ll}{3tt} = 40 \times 40 \times 20 \times \frac{3 \times 6 \times 13\frac{1}{3} \times 13\frac{1}{3} - 20 \times 20}{3 \times 66\frac{2}{3} \times 66\frac{2}{3}}.$ $= 24 \times 280 = 6720 = P$.

But the elliptic ſegment ABK is $= 54\frac{1}{2}$, to which adding the rectangle $AH = 20 \times 12 = 240$, makes $294\frac{1}{2}$ = the generating area $ABHG$; which being multiplied by $2OH = 8$, produces $2356 = Q$.

Then $\overline{P-Q} \times p = \overline{6720 - 2356} \times 3{\cdot}14159$ &c. $= 13709{\cdot}91033$ = the content required.

RULE II.

Divide 3 times the elliptic ſegment whoſe chord is the length of the caſk by the ſaid length, to the quotient add the leaſt or head diameter, and from the product ſubtract the greateſt or bung diameter, and multiply the remainder by 8 times the central diſtance; then take the product from the ſum of the ſquare of the head diameter and 2 times the ſquare of the bung diameter, multiply the difference by the length, and the product drawn into ·261799 &c. (= $\frac{1}{3}$ of ·785398 &c.) will be the content of the caſk.

That is, $\overline{2DD + dd - 8c \times \overline{-D+d+\frac{3S}{l}}} \times \frac{1}{3}nl =$ the content of the caſk, where $D = BE$ the bung diameter, $d = AF$ the head diameter, $c = HO$ the central diſtance, $l = GI$ the length, $s =$ the ſegment ABC, and $n = {\cdot}785398$ &c.—This is proved at corollary 3 to the laſt problem.

EXAMPLE.

The bung and head diameters, of a caſk which is the middle zone of an elliptic ſpindle, being 32 and 24, and its length 40 inches; required the content in ale and wine gallons, ſuppoſing the diſtance of the centers of the ellipſe and ſpindle to be 4 inches.

Here $D = 32$, $d = 24$, $l = 40$, $c = 4$, and the ſegment $ABC = s = 109$.

Then $\overline{2DD + dd - 8c \times \overline{-D + d + \frac{3s}{l}}} \times \frac{1}{3}nl = \overline{2624 - 5{\cdot}6} \times 40 \times {\cdot}261799$ &c. $= 27419{\cdot}8219$ cubic inches.

Which being divided by 282, and 231, we have 97·2334 ale gallons, and 118·7005 wine gallons, for the content required.

RULE III.

From the ſum of the ſquare of the leaſt or head diameter and 3 times the ſquare of the greateſt or bung diameter take 4 times the ſquare of the diameter equidiſtant from the two former, and from 4 times the ſaid middle diameter take the ſum of the ſaid leaſt and 3 times the greateſt diameter; then divide the former difference by the latter, and $\frac{1}{4}$ of the quotient will be the central diſtance, with which proceed as in the 2d rule.

That is, $\frac{1}{4} \times \frac{3DD + dd - 4mm}{-3D - d + 4m}$ = the central diſtance, *m* being the middle diameter, and *D* and *d* the other diameters as before.—By corollary 4 to the laſt problem.

EXAMPLE.

If the bung and head diameters be 32 and 24, and the length 40 inches, as in the laſt example; required the content, ſuppoſing the middle diameter to be $4\sqrt{91} - 8$.

Here $\frac{3DD + dd - 4mm}{-3D - d + 4m} \times \frac{1}{4} = \frac{3 \times 32^2 + 24^2 - 64 \times \overline{\sqrt{91} - 2}|^2}{-3 \times 32 - 24 + 16 \times \overline{\sqrt{91} - 2}}$ $\times \frac{1}{4} = \frac{-76 + 8\sqrt{91}}{-19 + 2\sqrt{91}} = 4 = c$ the central diſtance, the ſame as in the laſt example; and ſince the other parts are all the ſame, the content muſt likewiſe be the ſame.

PROBLEM XIX.

To find the content of the ſegment of an elliptic ſpindle.

RULE.

1. Multiply the ſquare of the altitude of the ſegment by the ſquare of the leſs axe of the ellipſe and divide the product by the ſquare of the greater axe, multiply the quotient by the difference between $\frac{1}{2}$ the length of the whole ſpindle, and $\frac{1}{3}$ of the altitude of the ſegment, and call the product *P*.

That is, $P = aacc \times \frac{\frac{1}{2}l - \frac{1}{3}a}{tt}$, where *c* is the conjugate and *t* the tranſverſe axe of the ellipſe, *a* is the altitude of the ſegment, and *l* the length of the whole ſpindle.

2. Multiply the double of the generating area by the diſtance of the centers of the ellipſe and ſpindle, and call the product $\mathcal{Q}$.

That is, $\mathcal{Q} = 2CA$, where $C = OH$, and $A =$ the area LAG.

3. Then the difference between P and $\mathcal{Q}$ drawn into 3·14159 &c. will be the content of the ſegment LAF of the ſpindle.

That is, $\overline{P-\mathcal{Q}} \times p = \overline{aacc \times \frac{\frac{1}{2}l - \frac{1}{3}a}{ll} - 2AC} \times p =$ the content of the ſegment.—By the demonſtration of problem 17.

EXAMPLE.

The axes of an ellipſe being 50 and 30, required the content of the ſegment of a ſpindle whoſe height is 10, the central diſtance being 9.

Here $t = 50$, $c = 30$, $a = 10$, and $C = 9$.

Then, by the nature of the ellipſe, as $30 : 50 :: (\sqrt{\overline{15+9} \times \overline{15-9}} = \sqrt{24 \times 6} =)\ 12 : \frac{5 \times 12}{3} = 20 = \frac{1}{2}l = LH$ the half of the length of the ſpindle.

And, by the ſame, as $50 : 30 :: (\sqrt{NP \times PM} = \sqrt{\overline{25+10} \times \overline{25-10}} = \sqrt{35 \times 15} =)\ 5\sqrt{21} : \frac{3 \times 5\sqrt{21}}{5} = 3\sqrt{21} = 13{\cdot}71495534 = PA$. Hence $BO - PA = 1{\cdot}285 = BK$.

But $\frac{KB}{2BO} = \frac{1{\cdot}285}{30} = {\cdot}0428\frac{1}{3}$, and $\frac{HB}{BO} = \frac{6}{30} = \frac{1}{5} = {\cdot}2$; the tabular numbers anſwering to which two quotients, in the table of circular ſegments, are ·01166678 and ·1118238; hence $50 \times 30 \times \overline{{\cdot}1118238 - {\cdot}01166678} = 1500 \times {\cdot}10015702 = 150{\cdot}23553 =$ the area $LB\mathcal{Q}$;

from the half of which taking the rectangle $AH = AG \times GH = 4{\cdot}714955 \times 10 = 47{\cdot}149$, leaves $27{\cdot}968765 = A$ for the generating area LAG.

Then $\overline{aacc \times \frac{\frac{2}{3}l - \frac{1}{3}d}{tt} - 2AC \times p} =$

$\overline{100 \times 900 \times \frac{20 - 3\frac{1}{3}}{2500} - 18 \times 27{\cdot}968765 \times 3{\cdot}14159 \text{ \&c.}} = \overline{36 \times \frac{50}{3} - 18 \times 27{\cdot}968765} \times 3{\cdot}14159 \text{ \&c.} = 18 \times 5{\cdot}364568 \times 3{\cdot}14159 \text{ \&c.} = 303{\cdot}35917 =$ the content of the ſegment LAF required.

PROBLEM XX.

To find the content of an univerſal ſpheroid, or a ſolid conceived to be generated by the revolution of a ſemi-ellipſe about its diameter, whether that diameter be one of the axes of the ellipſe or not.

RULE I.

Divide the ſquare of the product of the axes of the ellipſe by the axe of the ſolid or the diameter about which the ſemi-ellipſe is conceived to revolve, multiply the quotient by $\cdot 5236$, and the product will be the content required.

That is, $\frac{T^2 C^2}{d} \times \cdot 5236 =$ the content, T and C being the tranſverſe and conjugate axes of the ellipſe, and d the axe of the ſolid.

RULE II.

The continual product of $\cdot 5236$, the diameter about which the revolution is made, the ſquare of its conjugate diameter, and the ſquare of the ſine of the angle made by thoſe diameters (the radius being 1) will be the content.

That is, $dccss \times {}^{\cdot}5236 =$ the content, c being the conjugate diameter to d, and s the ſine of the angle made by the diameters.*

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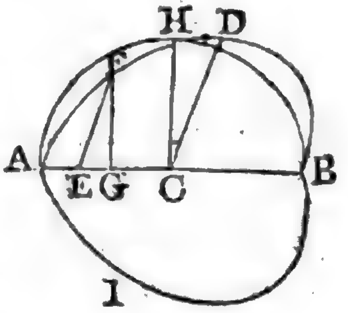
* DEMONSTRATION.



Let $ADBIA$ be a ſection of the ſolid through its axe AB; draw DC the ſemi-conjugate diameter to AB, as alſo the ordinate FE parallel thereto, and let fall the perpendicular FG.

Put T and C for the tranſverſe and conjugate axes of the ellipſe, $d =$ the diameter AB, or axe of the ſolid, $c = 2CD = \sqrt{T^2 + C^2 - d^2}$ its conjugate diameter, $a = \frac{TC}{dc} =$ ſine of the $\angle DCB$ or $\angle FEG$, and $b =$ its coſine, $x = AE$, and $p = 3{\cdot}14159$ &c.

Then $d : c :: \sqrt{dx - xx} : \frac{c\sqrt{dx - xx}}{d} = EF$, and

$\left\{\begin{array}{l} EF \times a = \frac{ac\sqrt{dx-xx}}{d} = FG \\ EF \times b = \frac{bc\sqrt{dx-xx}}{d} = GE \end{array}\right\}$, and hence $AG = GE + EA$ $= x + \frac{bc\sqrt{dx-xx}}{d}$; then $p \times FG^2 \times G\dot{A} = paac\dot{x} \times$ $\frac{dx - xx + \frac{\overline{\frac{1}{2}d - x}}{d} \times bc\sqrt{dx-xx}}{dd} =$ the fluxion of the ſolid, the fluent of which is $paacc \times \frac{\frac{1}{2}dx^2 - \frac{1}{3}x^3 + \frac{bc}{3d} \times \overline{dx - xx}|^{\frac{3}{2}}}{dd} =$ the meaſure of the part generated by AFG; and when $x = d$, it becomes $\frac{1}{6}pda^2c^2 = \frac{pT^2C^2}{6d}$ for the value of the whole ſolid.

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EXAMPLE I.

If the axes of an ellipſe be 50 and 30, and it be cut in two by a diameter whoſe length is 40; required the content of the ſolid generated by one of the halves about that diameter.

By rule 1, $\frac{50\times50\times30\times30}{40}\times{\cdot}5236=225000\times{\cdot}1309=29452{\cdot}5=$ the content required.

EXAMPLE II.

The diameter of an ellipſe about which it revolves being 40, its conjugate diameter $30\sqrt{2}$, and the ſine of the angle made by thoſe diameters $\frac{5}{8}\sqrt{2}$; required the content of the ſolid formed by the revolution of the ellipſe.

By rule 2, we have $40\times30\times30\times2\times\frac{5}{8}\times\frac{5}{8}\times2\times{\cdot}5236=225000\times{\cdot}1309=29452{\cdot}5=$ the content the ſame as before.

COROLLARY I.

If $d=T$, the rule becomes $\frac{1}{6}pTC^2$ for the oblong ſpheroid. And if $d=C$, it will be $\frac{1}{6}pCT^2$ for the oblate ſpheroid. Alſo if T, C, and d, be all equal, the rule will be $\frac{1}{6}pd^3$ for the ſphere. Which are the ſame with the rules before found for the ſame bodies.

COROLLARY II.

Draw CH perpendicular, and DH parallel to AB, and about the axes AB and $2CH$ deſcribe the ſemi-ellipſe AHB; then the ſpheroid generated by the revolution of the ſemi-ellipſe AHB about AB will be equal to the ſolid generated by the ſemi-ellipſe $AFDB$ about the ſame axe AB.

For $2CH=ac$, and therefore the ſolid $AFBI$ or $\frac{1}{6}pda^2c^2=\frac{1}{6}pd\times\overline{2CH}|^2=$ the ſpheroid whoſe axes are d and $2CH$.

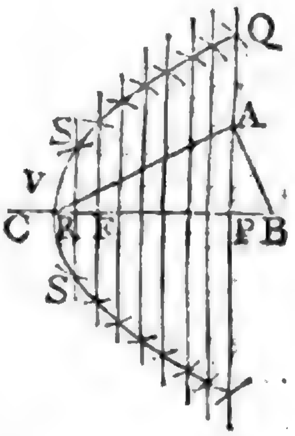
And ſo the ſolids generated by all ſemi-ellipſes upon the ſame baſe and between the ſame parallels, are all equal to each other.

SECTION IV.

Of parabolic Lines, Areas, Surfaces, and Solidities.

PROBLEM I.

To conſtruct a parabola; having given any ordinate PQ *to the axe, and its abſciſſa* VP.



1. FIND the focus F thus: Biſect PQ in A; draw AV, and AB perpendicular to it; take $VF=PB$, and F will be the focus.

Arithmetically.

Divide the ſquare of the ordinate by 4 times the abſciſſa, and the quotient will be the focal diſtance VF.

2. In the axe, produced without the vertex V, take $VC=VF$; draw ſeveral double ordinates SRS; with the radiuſes CR and center F deſcribe arcs cutting the correſponding ordinates in the points S.

Draw a curve through all the points of interſection, and it will be the parabola required.

PROBLEM II.

Of any abſciſſa x, *its ordinate* y, *and parameter* p; *having two given, to find the third.*

CASE I.

To find the parameter.

RULE.

Divide the ſquare of the ordinate by its abſciſſa, and the quotient will be the parameter. Or, take a third proportional to the abſciſſa and ordinate, for the parameter.

That is, $yy \div x = p$.

EXAMPLE.

If the abſciſſa be 9, and its ordinate 6; required the parameter.

Here $6 \times 6 \div 9 = 36 \div 9 = 4$ = the parameter.

CASE II.

To find the abſciſſa.

RULE.

Divide the ſquare of the ordinate by the parameter, and the quotient will be the abſciſſa.

That is, $yy \div p = x$.

EXAMPLE.

If the ordinate be 6, and the parameter 4; required the abſciſſa.

Here $6 \times 6 \div 4 = 36 \div 4 = 9$ = the abſciſſa.

CASE III.

To find the ordinate.

RULE.

Multiply the parameter by the abſciſſa, and the ſquare root of the product will be the ordinate.

That is, $\sqrt{px} = y$.

Ex-

EXAMPLE.

The abſciſſa being 9, and the parameter 4; required the ordinate.

Here $\sqrt{9 \times 4} = \sqrt{36} = 6 =$ the ordinate.

PROBLEM III.

Of any two abſciſſas A, B, *taken upon the ſame diameter, and their two ordinates* a, b; *having any three given to find the fourth.*

RULE.

The abſciſſas are to one another as the ſquares of their ordinates. That is, As any abſciſſa is to the ſquare of its ordinate, ſo is any other abſciſſa to the ſquare of its ordinate; and the contrary. Or, as the root of an abſciſſa is to its ordinate, ſo is the root of another abſciſſa, to its ordinate.

$$\text{Hence} \left\{ \begin{array}{l} \sqrt{A} : \sqrt{B} :: a : a\sqrt{\frac{B}{A}} = \frac{a\sqrt{AB}}{A} = b. \\ \sqrt{B} : \sqrt{A} :: b : b\sqrt{\frac{A}{B}} = \frac{b\sqrt{AB}}{B} = a. \end{array} \right.$$

$$\text{And} \left\{ \begin{array}{l} aa : bb :: A : \frac{Abb}{aa} = B. \\ bb : aa :: B : \frac{Baa}{bb} = A. \end{array} \right.$$

EXAMPLE I.

If an abſciſſa of 9 correſpond to an ordinate of 6, required the ordinate whoſe abſciſſa is 16.

Here $\sqrt{9} : \sqrt{16} :: 6 : 6\sqrt{\frac{16}{9}} = \frac{6 \times 4}{3} = 8 =$ the ordinate.

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EXAMPLE. II.

Required the abſciſſa correſponding to the ordinate 6, the ordinate belonging to the abſciſſa of 16 being 8.

Here $8\times8 : 6\times6 :: 16 : \frac{16\times6\times6}{8\times8} = 9 =$ the abſciſſa.

SCHOLIUM.

The demonſtration of the three preceding problems are omitted here, as they properly belong to, and are to be found in, all treatiſes of conics.

PROBLEM IV.

To find the length of the curve or arc of a parabola cut off by a double ordinate to the axe.

*RULE I.

Divide the ordinate by $\frac{1}{2}$ the parameter, and call the quotient q.

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* DEMONSTRATION.

Putting $z =$ any curve beginning at the vertex, $y =$ the ordinate to the axe at the extremity of the curve, $x =$ its abſciſſa, and $a = \frac{1}{2}$ the parameter of the axe, the equation to the curve is $2ax = yy$; hence $2a\dot{x} = 2y\dot{y}$, and $\dot{x}^2 = \frac{y^2\dot{y}^2}{aa}$; conſequently $\dot{z} = \sqrt{\dot{y}\dot{y} + \dot{x}\dot{x}} = \sqrt{\dot{y}^2 + \frac{y^2\dot{y}^2}{aa}} = \frac{\dot{y}\sqrt{aa+yy}}{a}$, and the corrected fluents give $z = \frac{y\sqrt{aa+yy}}{2a} + \frac{1}{2}a \times$ hyp. log. of $\frac{y+\sqrt{aa+yy}}{a} = \frac{1}{2}aq\sqrt{1+qq} + \frac{1}{2}a \times$ hyp. log. of $\overline{q+\sqrt{1+qq}}$, writing q for $\frac{y}{a}$. And the double of this quantity will give the value of the double curve as in the rule.

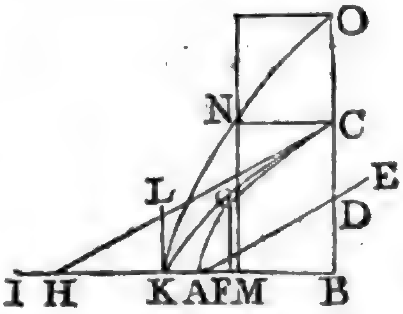
Add 1 to the ſquare of q, and call the root of the ſum s.

To the product of q and s add the hyperbolic logarithm of their ſum, and the laſt ſum multiplied by half the parameter will be the length of the whole curve on both ſides of the axe required.

That is, the curve $c = a \times \overline{qs + \text{hyp. log. of } q+s}$, where $q =$ the quotient of the ordinate divided by $\frac{1}{2}$ the parameter, $s = \sqrt{1+qq}$, and $a = \frac{1}{2}$ the parameter.

Note

COROLLARY I.



If for a be ſubſtituted its value $\frac{yy}{2x}$, there will be obtained $z = \sqrt{xx + \frac{1}{4}yy} + \frac{1}{2}a \times$ hyp. log. of $\frac{x + \sqrt{xx + \frac{1}{4}yy}}{\frac{1}{2}y}$. And this proves the truth of the conſtruction in COTES's *Harmonia Menſurarum page* 22, viz. that if F be the focus, and if AD, drawn to biſect BC in D, be produced till DE be $= AF \times$ hyp. log. of $\frac{BA + AD}{DB}$; then AE will be equal the curve AC.——For $\frac{1}{2}a = AF$, and $\sqrt{xx + \frac{1}{4}yy} = AD$.

COROLLARY II.

From ſect. 5 it will appear that the area of an hyperbola whoſe ſemi-tranſverſe axe is d, ſemi-conjugate a, and ordinate y, is $A = \frac{dy\sqrt{aa+yy}}{2a} - \frac{1}{2}ad \times$ hyp. log. of $\frac{y+\sqrt{aa+yy}}{a}$; but the parabolic curve is $C = \frac{y\sqrt{aa+yy}}{2a} + \frac{1}{2}a \times$ hyp. log. of $\frac{y+\sqrt{aa+yy}}{a}$; hence $\frac{A}{d} + C = \frac{y\sqrt{aa+yy}}{a} = \sqrt{y^2 + \frac{y^4}{aa}} = \sqrt{yy + 4xx} = 2AD = HC$ the tangent in the point C meeting the axe BA, produced, in H: Conſequently $C = HC - \frac{A}{d}$. Where the ſemi-tranſverſe d may be taken at pleaſure.

If

Note 1. If the common logarithm of any number be multiplied by 2·302585093, the product will be the hyperbolic logarithm of the ſame number.

2. If the value of s run into decimals, it will be much eaſieſt found by a trigonometrical canon; for s is the ſecant of the arc whoſe tangent is q, the radius being 1.

Ex-

If there be taken $d = y$, we ſhall obtain $C = HC - \frac{A}{y}$; and to find the diſtance of the ordinate y from the center of the hyperbola, we ſhall have $a : y\ (d) :: \sqrt{aa+yy} = \frac{y\sqrt{aa+yy}}{a}$ = the ſaid diſtance, and which, therefore, is $= HC$. Whence this conſtruction. —In the axe produced take $BI = HC$; with the center I, ſemi-tranſverſe IK = the ordinate BC, and ſemi-conjugate KL = the ſemi-parameter FG of the parabola, let be deſcribed the hyperbola KC, which will paſs through C; and let the rectangle $CBMN$ be equal the hyperbolic area KCB: Then will IM be equal the parabolic curve AC.

When the abſciſſa and tranſverſe axe of an hyperbola are given or conſtant, not only the ordinate but the area alſo is as the conjugate axe, and therefore the quotient ariſing from the diviſion of the area by the ordinate, is a conſtant quantity; and conſequently the parabolic curve $AC = C = HC - \frac{B}{z}$, where B is the area and z the ordinate of any hyperbola whoſe center is I, vertex K, and abſciſſa KB. From hence ariſes the following general conſtruction, given by Mr HUYGENS in his *Horolog. Oſcillat.* but without demonſtration. —With the center I and vertex K, as before, and any conjugate taken at pleaſure, deſcribe the hyperbola KO meeting BC, produced if neceſſary, in O; and, making the rectangle OM = the hyperbolic area KOB, MI will be equal to the parabolic curve AC as before.

COROLLARY III.

When $y = a = FG$, then $z = \frac{1}{2}a \times \overline{\sqrt{2} + \text{hyp. log. of } 1 + \sqrt{2}} = \frac{1}{2}a \times 2{\cdot}2955871 = 2{\cdot}2955871 \times AF = 1{\cdot}1477935 \times FG$ = the curve AG; F being the focus.

COROLLARY IV.

The lengths of ſimilar parts of parabolic curves will be as their parameters, or ordinates, or abſciſſas. For q will be the ſame in each.

EXAMPLE.

Required the length of the curve of a parabola cut off by a double ordinate, to the axe, whoſe length is 12, the abſciſſa being 2.

Here $x = 2$, and $y = 6$.

Hence $a = \frac{yy}{2x} = \frac{36}{4} = 9$, $q = \frac{y}{a} = \frac{6}{9} = \frac{2}{3}$, and $s = \sqrt{1+qq} = \sqrt{1+\frac{4}{9}} = \sqrt{\frac{13}{9}} = \frac{1}{3}\sqrt{13} = 1{\cdot}2018504$.—Or, by the table of tangents and ſecants, the ſecant correſponding to the tangent $\frac{2}{3}$ or ·666 &c. is $1{\cdot}2018504 = s$.

Then, $\frac{2}{3} + 1{\cdot}2018504 = 1{\cdot}868517$, whoſe common logarithm is ·271497, which being multiplied by 2·302585093 produces ·62514495 for its hyperbolic logarithm; alſo $\frac{2}{3} \times 1{\cdot}2018504 = {\cdot}8012336$; and the ſum of theſe two is ${\cdot}6251449 + {\cdot}8012366 = 1{\cdot}4263785$.

Wherefore $9 \times 1{\cdot}4263785 = 12{\cdot}8374065$ = the length of the curve required.

*RULE II.

Putting y to denote the ordinate, and q the quotient ariſing from the diviſion of the ordinate by half

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* DEMONSTRATION.

By the laſt, the fluxion of the curve was $\dot{z} = \dot{y}\sqrt{1+\frac{yy}{aa}}$ = (by extracting the root) $\dot{y} \times : 1 + \frac{y^2}{2a^2} - \frac{y^4}{2 \cdot 4a^4} + \frac{3y^6}{2 \cdot 4 \cdot 6a^6}$ &c. and, by taking the fluents and writing q for $\frac{y}{a}$, we obtain $z = y \times : 1 + \frac{q^2}{2 \cdot 3} - \frac{q^4}{2 \cdot 4 \cdot 5} + \frac{3q^6}{2 \cdot 4 \cdot 6 \cdot 7}$ &c. $= y \times : 1 + \frac{1 \cdot 1}{2 \cdot 3}q^2A - \frac{1 \cdot 3}{4 \cdot 5}q^2B + \frac{3 \cdot 5}{6 \cdot 7}q^2C$ &c. = the length of the curve from the vertex to the ordinate, the double of which will be that of the double curve. *Q.E.D.*

Co-

half the parameter, or from the diviſion of twice the abſciſſa by the ordinate; the length of the double curve will be denoted by the infinite ſeries $2y \times : 1 + \frac{q^2}{2.3} - \frac{q^4}{2.4.5} + \frac{3q^6}{2.4.6.7} - \frac{3.5q^8}{2.4.6.8.9}$ &c. or by $2y \times : 1 + \frac{1}{2.3}q^2A - \frac{1.3}{4.5}q^2B + \frac{3.5}{6.7}q^2C - \frac{5.7}{8.9}q^2D$ &c. where *A*, *B*, *C*, &c. denote the 1ſt, 2d, 3d, &c. terms.

Note, This ſeries will converge no longer than till $q = 1$, that is, when the ordinate to the curve, whoſe length is required, meets the axe in the focus; for if the ordinate *y* be beyond the focus, it will be greater than the ſemi-parameter, and conſequently *q* greater than 1.

EXAMPLE.

Let there be taken the ſame example as before, in which the abſciſſa is 2 and ordinate 6.

COROLLARY.

The hyperbolic logarithm of $q + \sqrt{1 + qq}$ is $= q \times : 1 - \frac{q^2}{2.3} + \frac{3q^4}{2.4.5} - \frac{3.5q^6}{2.4.6.7}$ &c. For, by the laſt rule, $\frac{2z}{a}$ — hyp. log. of $\overline{q + \sqrt{1 + qq}}$ is $= q\sqrt{1 + qq} = q \times : 1 + \frac{q^2}{2} - \frac{q^4}{2.4} + \frac{3q^5}{2.4.6}$ &c. and, by this rule, $\frac{2z}{a}$ is $= q \times : 1 + \frac{q^2}{3} - \frac{q^4}{4.5} + \frac{3q^6}{4.6.7}$ &c. but by taking the former of theſe from the latter, we have $q \times : 1 - \frac{q^2}{2.3} + \frac{3q^4}{2.4.5} - \frac{3.5q^6}{2.4.6.7} + \frac{3.5.7q^8}{2.4.6.8.9}$ &c. $=$ hyp. log. of $\overline{q + \sqrt{1 + qq}}$.

Then $\frac{2 \times 2}{6} = \frac{2}{3} = q$, which being uſed for it in the general ſeries, and the affirmative and negative terms collected, will appear as below:

$$\begin{array}{lr@{\qquad}lr}
A = & 1{\cdot}00000000 & C = & 0{\cdot}00493827 \\
B = & 0{\cdot}07407407 & E = & 16935 \\
D = & 78385 & G = & 1215 \\
F = & 4311 & I = & 117 \\
H = & 368 & L = & \underline{13} \\
K = & \underline{38} & & -0{\cdot}0051211
\end{array}$$

$+\ 1{\cdot}0749051$ ſum of the affirmative terms

$-\ 0{\cdot}0051211$ —— negative —

Dif. $1{\cdot}069784$ ſum of the ſeries

$12 = 2y$

$12{\cdot}837408$ = length of the curve nearly the ſame as before.

RULE III.

To the ſquare of the ordinate add $\frac{4}{3}$ of the ſquare of the abſciſſa, and the root of the ſum will be the length of the ſingle curve nearly, the double of which will be that of the curve on both ſides of the abſciſſa nearly.

That is, $\sqrt{yy + \frac{4}{3}xx} = c$ the length of the ſingle curve nearly, y being the ordinate and x the abſciſſa.*

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* DEMONSTRATION *of this and Rule* IV.

By the laſt rule, the curve is $c = y \times : 1 + \frac{1}{2.3}q^2 - \frac{1}{2.4.5}q^4 + \frac{3}{2.4.6.7}q^6$ &c. but $\sqrt{1 + \frac{1}{3}qq}$ $\quad 1 + \frac{1}{2.3}q^2 - \frac{1}{2.4.9}q^4 + \frac{3}{2.4.6.27}q^6$ &c.

EXAMPLE.

Taking again the ſame example, in which $x = 2$, and $y = 6$, we ſhall have $c = \sqrt{yy + \frac{4}{3}xx} = \sqrt{36 + \frac{16}{3}}$ $= 6{\cdot}4291 =$ the ſingle curve, the double of which is $12{\cdot}8582 =$ the length of the curve nearly.

RULE IV.*

To the ſquare of the ordinate add $\frac{2}{3}$ of the ſquare of the abſciſſa, and divide the ſum by the ordinate; then ſubtract 4 times this quotient from 9 times the length of the ſingle curve as found from the laſt rule, and $\frac{1}{5}$ of the remainder will be the length of the ſingle curve extremely near.

That is, $c = \overline{9\sqrt{yy + \frac{4}{3}xx} - 4 \times \frac{yy + \frac{2}{3}xx}{y}} \times \frac{1}{5}$ very nearly.

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&c. hence $\frac{c}{y} - \sqrt{1 + \frac{1}{3}qq} = -\frac{1}{2.5.9}q^4 + \frac{5}{4.7.27}q^6$ &c. and, ſuppoſing q not greater than 1 and rejecting the ſeries, c will be $= y\sqrt{1 + \frac{1}{3}qq} = \sqrt{yy + \frac{4}{3}xx}$ nearly. Which is rule 3.

Again, from the 1ſt ſeries, $\frac{c}{y} - 1 - \frac{1}{6}qq = -\frac{1}{2.4.5}q^4 + \frac{1}{4.4.7}q^6$ &c. or $\frac{4}{9} \times \overline{\frac{c}{y} - 1 - \frac{1}{6}qq} = -\frac{1}{2.5.9}q^4 + \frac{3}{4.7.27}q^6$ &c. and, from above, $\frac{c}{y} - \sqrt{1 + \frac{1}{3}qq} = -\frac{1}{2.5.9}q^4 + \frac{5}{4.7.27}q^6$ &c. hence, by ſubtracting, $\frac{c}{y} - \sqrt{1 + \frac{1}{3}qq} - \frac{4}{9} \times \overline{\frac{c}{y} - 1 - \frac{1}{6}qq} = \frac{1}{2.7.27}q^6$ &c. and, conſequently, the remaining ſeries being very ſmall, we ſhall obtain $c = \frac{1}{5}y \times \overline{9\sqrt{1 + \frac{1}{3}qq} - 4 \times \overline{1 + \frac{1}{6}qq}} = \frac{1}{5} \times \overline{9\sqrt{yy + \frac{4}{3}xx} - 4 \times \frac{yy + \frac{2}{3}xx}{y}}$ very nearly. Which is rule 4.*

And in this manner we may proceed to any degree of accuracy required.

EXAMPLE.

Taking, ſtill, the ſame example; we ſhall have $4 \times \frac{yy + \frac{2}{3}xx}{y} = 4 \times \frac{36 + \frac{8}{3}}{6} = 25\frac{7}{9}$; but by the laſt rule $c = 6{\cdot}4291$, hence $9 \times 6{\cdot}4291 = 57{\cdot}8619$; and $\frac{57{\cdot}8619 - 25\frac{7}{9}}{5} = 6{\cdot}4168 = c$, the double of which is $12{\cdot}8336$.

NOTE.

It muſt be obſerved, that, as theſe two approximations are derived from the 2d rule, they muſt be uſed only in ſuch caſes in which that might be applied, viz. thoſe in which the abſciſſa doth not exceed half the ordinate.

PROBLEM V.

To find the area included by the curve of a parabola and any right line, called the baſe of the ſegment or area.

RULE.*

Take $\frac{2}{3}$ of its circumſcribed parallelogram for the area required.

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* DEMONSTRATION.

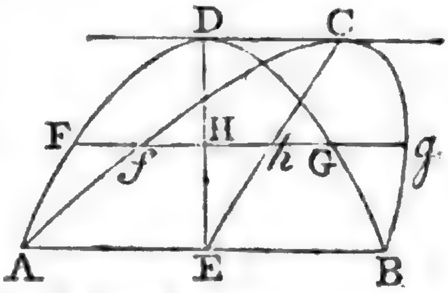
Put y = an ordinate to any diameter, x = its abſciſſa, and p = the parameter of that diameter; then the equation to the curve will be $px = yy$, and, putting s = the ſine of the angle made by the abſciſſa and ordinate, the fluxion of the area $\dot{a}$ will be $= sy\dot{x} = \frac{s^2\dot{y}}{}$; hence the area a is $= \frac{2sy^3}{3p} = \frac{2}{3}sxy$; that is, $\frac{2}{3}$ of the parallelogram having the ſame baſe and altitude.

NOTE.

The baſe of the circumſcribed parallelogram is the ſame with the baſe of the ſegment, their altitudes are, likewiſe, the ſame; and, therefore, the parabolic area will be equal to $\frac{2}{3}$ of its altitude multiplied by its baſe; that is $= \frac{2}{3}ab$; putting a to denote its altitude, and b its baſe.——It may, farther, be obſerved, that if the baſe be perpendicular to the diameter of the figure, then the altitude a will be the ſame with the abſciſſa of the figure; otherwiſe, the altitude is equal to the abſciſſa drawn into the natural ſine of the angle made by the abſciſſa and ordinate or baſe, the radius being 1. And this is to be obſerved in every other figure.

COROLLARY I.

Hence it is evident that all parabolas ADB, ACB, of equal baſes AB, and of equal altitudes or between the ſame parallels AB, DC, are equal to one another.



COROLLARY II.

Any common ſections FG, fg, of the parabolas ADB, ACB, are equal to each other.

For $(EC : Ch :: ED : DH ::)$ $AB^2 : fg^2 :: AB^2 : FG^2$; but $AB = AB$, therefore $fg = FG$.

COROLLARY III.

And hence, alſo, the ſegments FDG, fCg, are equal to each other.——For they are of equal baſes and altitudes.

COROLLARY IV.

Moreover, the fruſtums $AFGB$, $AfgB$, of equal ends and altitudes are equal to one another.

EXAMPLE I.

Required the area of a parabola, the abſciſſa being 2, and the ordinate, perpendicular to the abſciſſa, 6.

Here the altitude is 2, and the baſe or double ordinate is 12; therefore $\frac{2 \times 2 \times 12}{3} = 16 =$ the area required.

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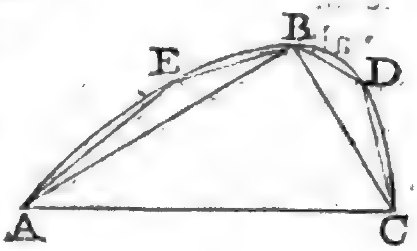
COROLLARY V.



Let *ABC* be a triangle having the ſame baſe and altitude with the parabolic ſegment *AEBDC*; then, becauſe the triangle is half of the circumſcribed parallelogram, it will be $\frac{3}{4}$ of the parabolic ſegment, and conſequently the ſegments *AEB*, *BDC*, together, will be $\frac{1}{4}$ of the whole portion *AEBDC*, or $\frac{1}{3}$ of the triangle *ABC*.

Again, it will appear that, if in the two ſegments *AEB*, *BDC*, be inſcribed triangles *AEB*, *BDC*, of the ſame baſes and altitudes with them, then theſe laſt triangles will be $\frac{3}{4}$ of their circumſcribed ſegments, and, conſequently, $\frac{1}{4}$ of the triangle *ABC*; if, in like manner, in the laſt made ſegments be inſcribed the greateſt triangles, they will be $\frac{1}{4}$ of the triangles immediately preceding them, or $\frac{1}{4 \cdot 4} = \frac{1}{16}$ of the firſt triangle; and ſo on continually: And conſequently all the inſcribed triangles, taken together, will be expreſſed by the ſeries $b \times : 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4}$ &c. where b denotes the firſt triangle; and which, when the ſeries is infinitely continued, will denote the area of the parabolic ſegment.

COROLLARY VI.

Hence the infinite ſeries $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3}$ &c. will be $= 1\frac{1}{3}$, as we alſo know from other principles; and, conſequently, the ſum of any finite number of terms of the ſeries $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3}$ &c. is leſs than $\frac{1}{3}$; and the ſum of the infinite number of them $= \frac{1}{3}$.

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EXAMPLE II.

If the baſe of a parabolic ſegment be 12, and its abſciſſa 2 make an angle of 30 degrees with it; what will be the area.

The ſine of 30° being half of radius, the altitude will be $2 \times \frac{1}{2} = 1$; and hence $\frac{2}{3} \times 12 = 8$ = the area.

PROBLEM VI.

To find the area of a fruſtum or zone of a parabola, included by two parallel right lines and the intercepted curves of the parabola.

RULE.

To one of the parallel ends add the quotient ariſing from the diviſion of the ſquare of the other by the ſum of the ſaid ends, multiply the ſum by the altitude of the fruſtum or diſtance of the ends, and $\frac{2}{3}$ of the product will be the area. Or divide the difference of the cubes of the diameters by the difference of their ſquares, and multiply the quotient by $\frac{2}{3}$ of the altitude.

That is, $\overline{D + \frac{dd}{D+d}} \times \frac{2}{3}a$, or $\overline{d + \frac{DD}{D+d}} \times \frac{2}{3}a$, or $\frac{D^3-d^3}{D^2-d^2} \times \frac{2}{3}a$ = the area, D, d being the two ends, and a the altitude.*

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* DEMONSTRATION.

By the property of the parabola, $DD-dd : a :: \left\{\begin{matrix} DD : \frac{aDD}{DD-dd} \\ dd : \frac{add}{DD-dd} \end{matrix}\right\}$ = the altitudes of two compleat ſegments whoſe baſes are the ends D, d

of

EXAMPLE I.

If the two parallel ends of the fruſtum of a parabola be 10 and 6, and the part of the abſciſſa, perpendicular to and connecting the middles of thoſe ends, be 4; what will be the area.

Here the abſciſſa, being perpendicular to the ends, will be the altitude of the figure, and therefore $\overline{10+\frac{6\times6}{10+6}}\times\frac{4\times2}{3}=\frac{196}{6}=32\frac{2}{3}$, or $\overline{6+\frac{10\times10}{10+6}}\times\frac{4\times2}{3}=\frac{196}{6}=32\frac{2}{3}=$ the area.

Ex-

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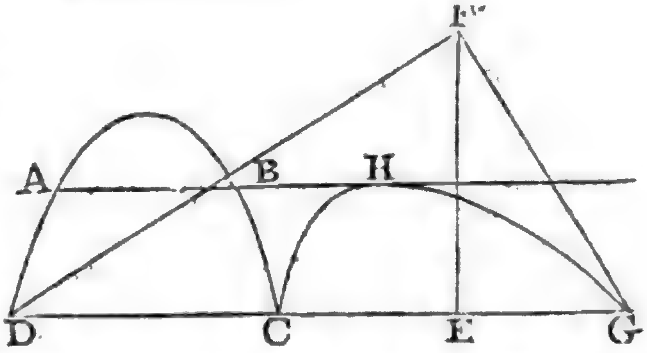
of the fruſtum; and, conſequently, the fruſtum or their difference is $\frac{2}{3}a\times\frac{D^3\backsim d^3}{DD\backsim dd}=\frac{2}{3}a\times\frac{DD+Dd+dd}{D+d}=\frac{2}{3}a\times\overline{D+\frac{dd}{D+d}}=\frac{2}{3}a\times\overline{d+\frac{DD}{D+d}}$. *Q.E.D.*

COROLLARY.



Hence a parabolic fruſtum is equal to a parabola of the ſame altitude, and whoſe baſe is equal to one end of the fruſtum encreaſed by a 3d proportional to the ſum of the ends and the other. Wherefore, in the one end, produced, of a parabolic fruſtum *ABCD*, take *CE* = the other end *AB*; raiſe *EF* perpendicular to *EC* and = *CD*; draw *DF*, and perpendicular to it *FG* meeting with *CE*, produced, in *G*: Then if upon the baſe *CG* be deſcribed a parabola *CHG* touching the line *ABH*, it will be equal to the fruſtum *ABCD*.

For $CG=CE+EG=AB+EG=AB+\frac{FE^2}{ED}=AB+\frac{DC^2}{DC+Ab}$.

EXAMPLE II.

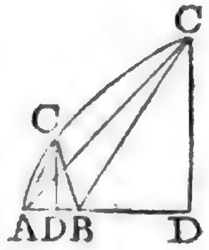
If the part of the abſciſſa, connecting the middles of the two ends, make with them an angle of 48° $35\frac{5}{12}'$; required the area, the other dimenſions being as in the firſt example.

The ſine of 48° $35\frac{5}{12}'$ being ·75 or $\frac{3}{4}$ very nearly, hence $4 \times \frac{3}{4} = 3$ = the altitude = $\frac{3}{4}$ of that in the laſt example, and therefore the area here muſt be $\frac{3}{4}$ of that above; that is, $32\frac{2}{3} \times \frac{3}{4} = \frac{98}{4} = 24\frac{1}{2}$ = the area required.

PROBLEM VII.

To find the area included by the focal diſtance, the line drawn from the focus to the curve, and the contained arc of the parabola.

RULE.



Upon the axe, or focal diſtance *AB*, produced if neceſſary, having demitted the ordinate or perpendicular *CD*, cutting off the abſciſſa *DA*; then

To the focal diſtance *BA* add $\frac{1}{3}$ of the abſciſſa *AD*, multiply the ſum by the ordinate *DC*, and $\frac{1}{2}$ of the product will be the area of the part *ACB*.

* That is, $\overline{BA + \frac{1}{3}AD} \times \frac{1}{2}DC$ = the area.

Note, The focal diſtance *AB* is $\frac{1}{4}$ of the parameter.

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* DEMONSTRATION.

Putting $AD = x$, $DC = y$, and $BA = a$; $ACB = ACD \pm DCB$ will be $= \frac{2}{3}xy \pm \frac{1}{2}y \times \overline{\mp x \pm a} = \frac{2}{3}xy - \frac{1}{2}xy + \frac{1}{2}ay = \frac{1}{2}y \times \overline{\frac{1}{3}x + a}$.
Q.E.D.

EXAMPLE.

If the abſciſſa be 2, and the ordinate 6; required the area of ACB.

By caſe 1 prob. 2, we have $2 : 6 :: 6 : 18$ = the parameter, and hence $\frac{18}{4} = 4\frac{1}{2} = AB$.

Then $\overline{4\frac{1}{2}+\frac{2}{3}}\times 3 = 5\frac{1}{6}\times 3 = 15\frac{1}{2}$ = the area required.

RULE II.

ſubtract the focal diſtance, or diſtance between the focus and the beginning of the arc, from the diſtance between the focus and the end of the arc, and multiply the remainder by the ſaid focal diſtance; then multiply the root of the product by the ſum of the diſtance between the end of the arc and focus and double the focal diſtance, and $\frac{1}{3}$ of this product will be the area.

* That is, $\frac{1}{3}\sqrt{AB\times\overline{CB-BA}}\times\overline{2AB+BC}$ = the area ACB.

EXAMPLE.

Taking here the ſame example as before, we have $AB = 4\frac{1}{2}$, and $BC = DA + AB = 2 + 4\frac{1}{2} = 6\frac{1}{2}$; and hence $\frac{1}{3}\sqrt{4\frac{1}{2}\times 2}\times\overline{9+6\frac{1}{2}} = 15\frac{1}{2}$ = the area as before.

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* DEMONSTRATION.

By the nature of the parabola, $z = CB$ is $= BA + AD = a + x$, and $x = z - a$, alſo $\frac{1}{2}y = \sqrt{ax} = \sqrt{a\times\overline{z-a}}$; which values of x and y, written for them in the laſt rule, give $\frac{1}{3}\sqrt{a\times\overline{z-a}}\times\overline{2a+z}$ = the area ACB. Q.E.D.

PROBLEM VIII.

To find the curve ſurface of a paraboloid.

RULE I.*

To the ſquare of the ordinate, or ſemi-diameter of the baſe, add 4 times that of the axe, and the root of

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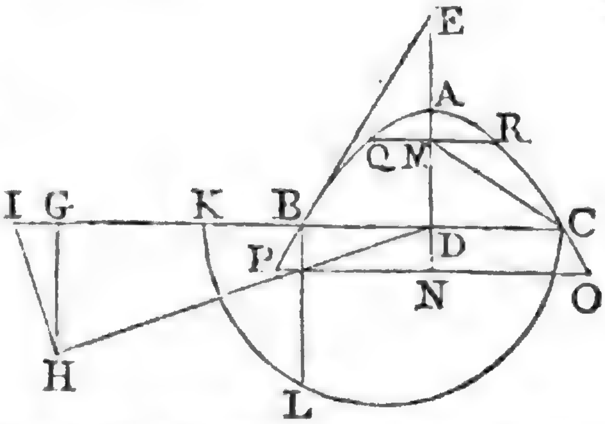
* DEMONSTRATION.

Calling the ordinate y, the abſciſſa x, the curve z, and the parameter p; alſo $a = 3{\cdot}14159$ &c. the equation of the generating parabola will be $px = yy$; hence $p\dot{x} = 2y\dot{y}$, and the fluxion of the ſurface $\dot{s} = 2ay\dot{z} = 2ay\sqrt{\dot{y}^2 + \dot{x}^2} = 2ay\sqrt{\dot{y}^2 + \frac{4y^2\dot{y}^2}{pp}} = \frac{4ay\dot{y}}{p}\sqrt{\tfrac{1}{4}pp + yy}$; and, by taking the correct fluents, the ſurface s will be $= \frac{4a}{3p} \times \overline{\tfrac{1}{4}pp + yy}|^{\frac{3}{2}} - \frac{1}{6}app = \frac{2}{3}ay \times \frac{\overline{yy + 4xx}|^{\frac{3}{2}} - y^3}{4xx} = \frac{2}{3}ay \times \frac{\overline{yy + 4xx}|^{\frac{3}{2}} - y^3}{\overline{yy + 4xx}|^{\frac{2}{2}} - y^2} = \frac{2}{3}ay \times \frac{t^3 - y^3}{t^2 - y^2} = \frac{2}{3}ay \times \frac{tt + ty + yy}{t + y} = \frac{2}{3}ay \times \overline{t + \frac{yy}{t + y}} = \frac{2}{3}ay \times \overline{y + \frac{tt}{t + y}}$, putting $t = \sqrt{yy + 4xx} = BE$ the tangent to the curve at the extremity of the ordinate and intercepted by the axe produced, *Q.E.D.*

COROLLARY I.

In CB, produced, take $BG =$ the tangent BE; erect GH perpendicular to GB and equal to BD; draw DH, and perpendicular thereto HI meeting BG, produced in I; take $BK = \frac{1}{3}$ of BI; and upon KC deſcribe a circle meeting in L with BL perpendicular to BC: And the circle whoſe radius is BL will be equal to the curve ſurface of the paraboloid BAC. For,



of the ſum will be the tangent of the curve at the baſe and intercepted by the axe produced; let this tangent be called t, viz. $t = \sqrt{yy + 4xx}$, x being the axe, and y the ordinate.

Then, by problem 6, find the area of the fruſtum of a parabola whoſe parallel ends are t and y, and its altitude $= y$.

And this area multiplied by 3·14159 &c. will be the curve ſurface of the paraboloid ABC required.

EXAMPLE.

Required the curve ſurface of a paraboloid whoſe axe is 20, and the diameter of its baſe 60.

Here

For, by the conſtruction, $BL^2 = KB \times BC = CB \times \frac{1}{3}BI = \frac{2}{3}BD \times \overline{BG + GI} = \frac{2}{3}BD \times BE + \frac{HG^2}{GD} = \frac{2}{3}BD \times BE + \frac{BD^2}{EB + BD} = \frac{2}{3}y \times \overline{t + \frac{yy}{t+y}}$.

COROLLARY II.

The curve ſurface of the paraboloid is to the area of its circular baſe, as IK is to BD.

For the baſe is $= a \times BD^2$, and the curve ſurface $= \frac{2}{3}IB \times BD \times a = a \times IK \times BD$.

COROLLARY III.

If M be the focus of the generating parabola, and there be taken $AN = MC = DA + AM$, and the double ordinates PNO, QMR be drawn; the ſurface generated by BAC will be to the area $OPQR$, as $a = 3{\cdot}14159$ &c. to 1, or as the circumference of a circle to its diameter.

For the ſurface is $= \frac{4a}{3p} \times \overline{\frac{1}{4}pp + yy}|^{\frac{3}{2}} - \frac{1}{6}app = \frac{4a}{3p} \times \overline{\frac{1}{4}pp + px}|^{\frac{3}{2}} - \frac{1}{6}pp = \frac{4}{3}a \times \overline{\frac{1}{4}p + x}\sqrt{\frac{1}{4}pp + px} - \frac{1}{6}app = \frac{4}{3}a \times MC\sqrt{MC \times p} - \frac{1}{6}app = \frac{4}{3}a \times AN \times NO - \frac{4}{3}a \times AM \times MR = a \times \overline{PAO - QAR} = a \times OPQR$.

Here $\sqrt{30^2+40^2} = 50$ is the tangent. Then, by prob. 5, $\overline{50+\frac{30\times30}{50+30}} \times \frac{2}{3} \times 30 \times 3{\cdot}14159$ &c. $= \frac{4900}{80} \times 20 \times 3{\cdot}14159$ &c. $= 1225 \times 3{\cdot}14159$ &c. $= 3848{\cdot}451$ = the ſurface required.

*RULE II.

Divide the difference between the cube of the tangent and the cube of the ordinate by 4 times the ſquare of the axe, multiply the quotient by $\frac{2}{3}$ of the ordinate, and the product multiplied by 3·14159 &c. will be the ſurface.

That is, $\frac{t^3-y^3}{4xx} \times \frac{2}{3}cy$ = the ſurface; where $t = \sqrt{yy+4xx}$ = the tangent, x = the abſciſſa or axe, y = the ordinate or ſemi-diameter of the baſe, and $c = 3{\cdot}14159$ &c.

EXAMPLE.

Taking the ſame example as in the laſt, in which $x = 20$, $y = 30$, and therefore $t = \sqrt{40^2+30^2} = 50$; we ſhall have $\frac{50^3-30^3}{40\times40} \times 30 \times \frac{2}{3}c = \frac{980}{16} \times 20c = 1225 \times 3{\cdot}14159$ &c. $= 3848{\cdot}451$ = the ſurface as before.

PROBLEM IX.

To find the curve ſurface of the fruſtum of a paraboloid, having given its altitude and the diameters of its ends.

RULE.

Divide the difference of the ſquares of the ſemi-diameters of the ends by their diſtance or by the al-

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* This rule appears in the inveſtigation of the laſt.

titude of the fruſtum, or divide the difference of the ſquares of the whole diameters by 4 times the altitude, and the quotient will be the parameter of the axe of the generating parabola.

Find the two ſums ariſing from the addition of the ſquare of the parameter and the ſquare of each diameter; multiply each ſum into its root, and divide the difference of the products by the parameter; multiply the quotient by 3·14159 &c. and $\frac{1}{6}$ of the product will be the ſurface of the fruſtum.

* That is, $\frac{\overline{pp+DD}|^{\frac{3}{2}} - \overline{pp+dd}|^{\frac{3}{2}}}{p} \times \frac{1}{6}c =$ the ſurface, D and d being the diameters of the ends, $p =$ the parameter $= \frac{DD-dd}{4a}$, $a =$ the altitude, and $c =$ 3·14159 &c.

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* DEMONSTRATION.

In the inveſtigation of rule 1 of the laſt problem, it appears that $\frac{4c}{3p} \times \overline{\tfrac{1}{4}pp+yy}|^{\frac{3}{2}} - \frac{1}{6}cpp$ or $\frac{c}{6p} \times \overline{pp+DD}|^{\frac{3}{2}} - \frac{1}{6}cpp$ is the ſurface of the ſegment whoſe baſe diameter is D, and $\frac{c}{6p} \times \overline{pp+dd}|^{\frac{3}{2}} - \frac{1}{6}cpp$ = that whoſe diameter is d; and, by taking the difference, $\frac{1}{6}c \times \frac{\overline{pp+DD}|^{\frac{3}{2}} - \overline{pp+dd}|^{\frac{3}{2}}}{p}$ will expreſs the ſurface of the fruſtum the diameters of whoſe ends are D, d.

And that the parameter is $= \frac{DD-dd}{4a}$, appears thus: Since $pX = YY$, and $px = yy$, therefore $pX - px = YY - yy$, and $p = \frac{YY-yy}{X-x} = \frac{DD-dd}{4a}$.

EXAMPLE.

Required the curve ſurface of the fruſtum of a paraboloid, the altitude being $25\frac{79}{80}$ and the diameters of its ends 48 and 15.

Here $D = 48$, $d = 15$, and $a = 25\frac{79}{80}$.

Therefore $\frac{DD - dd}{4a} = \frac{48^2 - 15^2}{4 \times 25\frac{79}{80}} = \frac{2079}{103\frac{19}{20}} = 20 = p$.

Then $\sqrt{pp + DD} = \sqrt{20^2 + 48^2} = \sqrt{2704} = 52$.

And $\sqrt{pp + dd} = \sqrt{20^2 + 15^2} = \sqrt{625} = 25$.

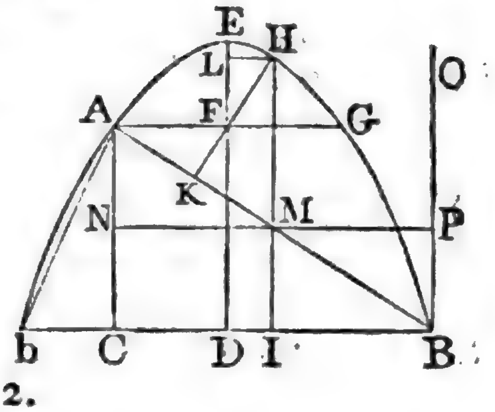
Hence $\frac{2704 \times 52 - 625 \times 25}{20 \times 6} \times c = \frac{41661}{40} \times 3{\cdot}14159$ &c. $= 4166{\cdot}1 \times {\cdot}785398$ &c. $= 3272{\cdot}0472885$ = the ſurface required.

PROBLEM X.

If a paraboloid be cut by a plane, oblique to its axe, it is required to find the axes of the elliptic ſection.

RULE.

It is evident that AB is the tranſverſe axe. And if Bb be a double ordinate to the axe and meeting the diameter AC in C, then BC will be the other or conjugate axe of the ſection. As is proved in corollary 7 to prop. 1 ſect. 2.



EXAMPLE.

If a paraboloid whoſe axe is $45\frac{5}{7}$, and its baſe or greateſt double ordinate 32, be cut by a plane paſſing through the extremity of the baſe and the oppoſite ſide of the figure at the height of 20 above the baſe; what are the axes of the ſection.

Draw AF perpendicular to and meeting with the axe DE in F.——Then $FE = ED - DF = 45\frac{5}{7} - 20 = 25\frac{5}{7}$; and, by the nature of the parabola, $DE : EF :: DB^2 : FA^2 = \frac{DB^2 \times FE}{ED} = \frac{16^2 \times 25\frac{5}{7}}{45\frac{5}{7}} = \frac{16^2 \times 5\frac{1}{7}}{9\frac{1}{7}} = \frac{16^2 \times 36}{64} = 4 \times 36 = 144$; and hence $FA = DC = \sqrt{144} = 12$: wherefore $CB = BD + DC = 16 + 12 = 28 =$ the conjugate axe. And $AB = \sqrt{BC^2 + CA^2} = \sqrt{28^2 + 20^2} = 4\sqrt{7^2 + 5^2} = 4\sqrt{74} = 34{\cdot}4093010682 =$ the tranſverſe axe required.

EXAMPLE II.

Let the ſame paraboloid be cut through the extremity of the baſe by a plane cutting the figure again on the ſame ſide of the axe, and at the ſame diſtance of 20 above the baſe; to find the axes of the ſection.

Here $Cb = bD - DC = 16 - 12 = 4 =$ the conjugate axe. And $bA = \sqrt{AC^2 + Cb^2} = \sqrt{20^2 + 4^2} = 4\sqrt{5^2 + 1^2} = 4\sqrt{26} = 20{\cdot}39607804 =$ the tranſverſe.

PROBLEM XI.

To find the ſolidity of any ſegment of a paraboloid, whoſe baſe is either perpendicular or oblique to the axe.

*RULE I.

For both right and oblique ſegments.

Multiply the baſe by half the altitude, and the product will be the content. NOTE.

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* DEMONSTRATION.

Let b denote the baſe, a the altitude, $x =$ any variable abſciſſa, or part of the diameter drawn through the middle of the baſe, and $s =$

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NOTE.

When the baſe is perpendicular to the axe, it is a circle, and the altitude is equal to the whole length of the axe contained between the baſe and the vertex.—But if the baſe be oblique to the axe, it will be an ellipſe, and the altitude will be the perpendicular

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$s =$ the ſine of the angle formed by this diameter and its ordinate, to the radius 1.

Then as $a : sx :: b : \frac{bsx}{a} =$ the ſection correſponding to the abſciſſa x, by prop. 1 ſect. 2. Wherefore $\frac{sx}{a} \times bs\dot{x} =$ the fluxion of the ſolid; and, by taking the fluent, the ſolid itſelf will be $\frac{bssxx}{2a}$; which, when $sx = a$, becomes $\frac{1}{2}ab$ for the whole ſolid whoſe baſe is b and and altitude a. *Q.E.D.*

COROLLARY I.

A paraboloid is equal to half a priſm of the ſame baſe and altitude; or a priſm, a ſemi-ſpheroid or ſemi-ſphere, a paraboloid, and a pyramid, all of equal baſes and altitudes, are to one another as the numbers 1, $\frac{2}{3}$, $\frac{1}{2}$, and $\frac{1}{3}$, or as 6, 4, 3, 2; and are, therefore, in a diſcontinued geometric proportion whoſe ratio is that of 3 to 2. Which will appear by comparing the above value of the paraboloid with thoſe of the other ſolids.——When the baſe of the paraboloid is perpendicular to the axe, or when the paraboloid is right, the ſemi-ſpheroid or ſemi-ſphere will, alſo, be right, and the priſm and pyramid will be the right or common cylinder and cone, the common baſe being a circle; but if the paraboloid be oblique by having its baſe oblique to its axe, the common baſe of all the ſolids will be an ellipſe, the ſemi-ſpheroid will, alſo, be oblique by having its baſe oblique to its axe, and the priſm and pyramid may be either right or oblique.

COROLLARY II.

If upon the diameter $NP = BC =$ the conjugate axe of the ſection AB be deſcribed a circle, the oblique paraboloid or ſegment AHB will cular demitted to the baſe from the vertex of the diameter drawn through the middle of the baſe, or it will be equal to the product ariſing from the multiplication of the part of the ſaid diameter, intercepted by the vertex and the baſe, by the ſine of the angle of inclination to the baſe.

Ex-

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will be equal to the right paraboloid whoſe baſe is that circle, and its altitude MH. For, the diameter of the circle being the conjugate axe of the baſe of the oblique paraboloid, the circle will be to that baſe as the conjugate is to the tranſverſe axe, that is, as CB to BA; but KH is to HM in the ſame proportion of CB to BA; conſequently $KH \times$ elliptic baſe AB will be $= HM \times$ circular baſe NP or BC.

It is alſo evident, that all ſegments, right or oblique, having the ſame vertex H, and the tranſverſe axes of whoſe baſes paſs through the ſame point M and terminate in AC, OP, parallel to and equidiſtant from HM, will be equal to one another.

COROLLARY III.

If B, b denote the two baſes or parallel ends of the fruſtum of a paraboloid, either right or oblique, and d the diſtance of the ends, or the altitude of the fruſtum; then $\frac{1}{2}d \times \overline{B+b}$ will be the ſolidity of the fruſtum.

For, by this problem, $\frac{1}{2}AB$ is the ſegment whoſe baſe is B and altitude A, and $\frac{1}{2}ab$ is that whoſe baſe is b and altitude a; wherefore the fruſtum, or the difference of the ſegments, is $\frac{1}{2}AB - \frac{1}{2}ab$: But, by the nature of the paraboloid, $B-b : d\ (A-a) :: B : A = \frac{Bd}{B-b}$, and $B-b : d :: b : a = \frac{bd}{B-b}$; which values of A and a being ſubſtituted for them, we obtain $\frac{1}{2}AB - \frac{1}{2}ab = \frac{dB^2 - db^2}{2B - 2b} = \frac{1}{2}d \times \overline{B+b}$.

COROLLARY IV.

Let $AGBb$ be a right fruſtum, HI the diameter to the double ordinate AB, and HK perpendicular to AB, that is, HK the altitude of the oblique ſegment AHB. Then the value of the oblique ſegment AHB is $\frac{1}{2}n \times AB \times BC \times HK$, putting $n = \cdot785398$ &c.

But

EXAMPLE I.

Required the ſolidity of a right paraboloid whoſe axe is 30 and diameter of its baſe 40.

By the rule, $40^2 \times 15 \times \cdot 785398$ &c. $= 18849\cdot 5559215$ = the ſolidity required.

Ex-

But, by the property of the figure, $BD^2 - DC^2 : CA :: BD^2 - DI^2 : \frac{BD^2 - DI^2}{BD^2 - DC^2} \times CA = IH$; hence $MH = HI - IM = HI - \frac{1}{2}CA = \frac{BD + DC}{BD - DC} \times \frac{1}{4}CA = \frac{BC \times CA}{8DI}$; and, by ſimilar triangles, $AB : BC :: MH : HK = \frac{HM \times CB}{BA} = \frac{BC^2 \times CA}{AB \times 8DI}$; therefore the oblique ſegment AHB, or $\frac{1}{2}n \times AB \times BC \times HK$, will be $= \frac{BC^3}{8DI} \times \frac{1}{2}CA \times n = \frac{BC^4}{Bb^2 - AG^2} \times CA \times \frac{1}{2}n$.

Or, by ſubſtituting the value of CA, viz. $\frac{BD^2 - DC^2}{BD^2} \times DE = \frac{BC \times 2DI}{BD^2} \times DE$, inſtead of it, the ſame ſegment AHB will be $= \frac{BC^4}{8BD^2} \times DE \times n = \frac{BC^4}{Bb^2} \times DB \times \frac{1}{2}n$.

COROLLARY V.

If from $DE \times Bb^2 \times \frac{1}{2}n$, the value of the right ſegment bEB, be taken the above value of the oblique ſegment AHB, there will remain $\frac{Bb^4 - BC^4}{Bb^2} \times DE \times \frac{1}{2}n$, or $\frac{Bb^4 - BC^4}{Bb^2 - AG^2} \times CA \times \frac{1}{2}n$ for the value of the greater ungula bAB.

COROLLARY VI.

And if from the ſaid value of the oblique ſegment AHB, be taken that of the right ſegment AEG, viz. $\frac{AG^4}{Bb^2} \times DE \times \frac{1}{2}n$ or $\frac{AG^4}{Bb^2 - AG^2} \times CA \times \frac{1}{2}n$, there will remain $\frac{BC^4 - AG^4}{Bb^2} \times DE \times \frac{1}{2}n$ or $\frac{BC^4 - AG^4}{Bb^2 - AG^2} \times CA \times \frac{1}{2}n$ for that of the leſs ungula AGB.

EXAMPLE II.

If from the above right paraboloid be cut a part BAb, by a plane paſſing through the extremity of the baſe, and the oppoſite ſide at the height AC of $22\frac{1}{2}$ above the baſe; it is required to find the content of the oblique ſegment AHB.

Here $FE = ED - CA = 30 - 22\frac{1}{2} = 7\frac{1}{2}$; then, by the property of the parabola, $\sqrt{DE} : \sqrt{EF} :: BD : AF = BD\sqrt{\frac{FE}{ED}} = 20\sqrt{\frac{7\frac{1}{2}}{30}} = 20\sqrt{\frac{1}{4}} = 20 \times \frac{1}{2} = 10$; hence $BD + DC = BD + AF = 20 + 10 = 30 = CB$ the conjugate diameter of the baſe, by the laſt problem; and $\sqrt{AC^2 + CB^2} = \sqrt{22{\cdot}5^2 + 30^2} = 7{\cdot}5\sqrt{3^2 + 4^2} = 7{\cdot}5 \times 5 = 37{\cdot}5 = BA$ the tranſverſe. Now H being the vertex of the diameter HMI drawn through M the middle of AB, and HK perpendicular to AB, the triangles HKM, MIB, BAC, will be ſimilar; but $MB = \frac{1}{2}BA$, and therefore $MI = \frac{1}{2}CA = 11\frac{1}{4}$; but $HL = DI = \frac{BD - DC}{2} = \frac{20 - 10}{2} = 5$; then, by the property of the parabola, $BD^2 : BD^2 - LH^2 :: DE : HI = \frac{BD^2 - LH^2}{BD^2} \times DE = \frac{20^2 - 5^2}{20^2} \times 30 = \frac{4^2 - 1^2}{4^2} \times 30 = \frac{15^2}{8} = \frac{225}{8} = 28\frac{1}{8}$; hence $HI - IM = 28\frac{1}{8} - 11\frac{1}{4} = 16\frac{7}{8} = MH$; and, by ſimilar triangles, $AB : BC :: MH : HK = \frac{MH \times CB}{BA} = \frac{16\frac{7}{8} \times 30}{37{\cdot}5} = 13\frac{1}{2} =$ the altitude.

Then $CB \times BA \times \frac{1}{2}HK \times {\cdot}785398$ &c. $= 30 \times 37{\cdot}5 \times 6{\cdot}75 \times {\cdot}785398$ &c. $= 7593{\cdot}75 \times {\cdot}785398$ &c. $= 5964{\cdot}1173033 =$ the ſegment AHB required.

RULE

RULE II.

For the oblique ſegment.

Divide the fourth power of BC by the ſquare of Bb, multiply the quotient by $\frac{1}{2}DE$, and the product multiplied by ·785398 &c. will be the content of the oblique ſegment AHB.

That is, $\frac{BC^4}{Bb^2} \times \frac{1}{2}DE \times n = AHB$. By corollary 4.

EXAMPLE.

Taking here the laſt example, we have $\frac{BC^4}{Bb^2} \times \frac{1}{2}DE \times n = \frac{30^4}{40^2} \times 15 \times {\cdot}785398$ &c. $= \frac{30^2 \times 9 \times 15}{16} \times {\cdot}785398$ &c. $= 759375 \times {\cdot}785398$ &c. $= 5964{\cdot}1173033$ the ſame as before.

PROBLEM XII.

To find the content of the fruſtum of a paraboloid, having its ends parallel to each other, and either right or oblique to the axe.

RULE.

Multiply the ſum of the two ends by their diſtance or altitude of the fruſtum, and half the product will be the content.—As is proved in corollary 3 to the laſt problem.

NOTE.

The ſame things may be obſerved here as in the note to the laſt problem.

EXAMPLE.

The length of a caſk, compoſed of two equal fruſtums of a paraboloid, is 45 inches; required the content in ale and wine gallons, ſuppoſing the bung diameter to be 40 and the head diameter 20 inches.

Here $\overline{40^2+20^2} \times {\cdot}785398$ &c. $\times 22\frac{1}{2} = 45000 \times {\cdot}785398$ &c. $= 35342{\cdot}917352885$ = the content in inches.

Then $\frac{35342{\cdot}91735}{282} = 125{\cdot}329494$ ale gallons.

And $\frac{35342{\cdot}91735}{231} = 152{\cdot}99964$ wine gallons.

PROBLEM XIII.

To find the ſolidity of the parabolic ungula bAB, *or* BAG, *made by a plane paſſing through the oppoſite extremities of the ends of the fruſtum* AGBb.

RULE.

Divide the difference of the 4th powers of the diameter of the baſe, and half the ſum of the diameters of the ends of the fruſtum, by the difference of the ſquares of the ſaid diameters; multiply the quotient by ·785398 &c. and the product multiplied by half the altitude will produce the content.

That is, $\frac{Bb^4 - BC^4}{Bb^2 - AG^2} \times CA \times \frac{1}{2}n$ = the ungula bAB,

And $\frac{BC^4 - AG^4}{Bb^2 - AG^2} \times CA \times \frac{1}{2}n$ = the ungula BAG.— As is proved in corol. 4 and 5 of prob. 10.

EXAMPLE I.

If a veſſel, in form of the fruſtum of a paraboloid, and open at the narrow end, be ſo placed, that the liquor in it may juſt cover the bottom, whoſe diameter is 40 inches, and riſe to the height of 22½ inches towards the top of it; required the quantity of liquor contained in the veſſel, ſuppoſing the diameter of the veſſel at the upper edge of the liquor to be 20 inches.

Here $Bb = 40$, $CA = 22\frac{1}{2}$, and $AG = 20$; conſequently $BC = \frac{40+20}{2} = 30$.

Hence $\frac{40^4 - 30^4}{40^2 - 20^2} \times 22\frac{1}{2} \times \frac{{\cdot}785398 \text{ \&c.}}{2} = \frac{17500}{96} \times 90 \times {\cdot}785398$ &c. $= 12885{\cdot}438618$ cubic inches; which being divided by 282 or by 231, will give $45{\cdot}693044$ ale, or $55{\cdot}781119$ wine gallons.

EXAMPLE II.

If the veſſel have its dimenſions inverted, that is, if AGB be the part filled, AB being the ſurface of the liquor, the bottom diameter AG being 20, the diameter Bb at the top of the liquor 40, and the altitude AC $22\frac{1}{2}$ inches; required the quantity of liquor.

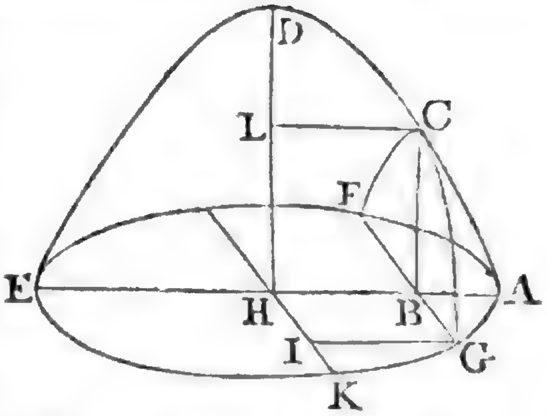
Here $\frac{BC^4 - AG^4}{Bb^2 - AG^2} \times AC \times \frac{1}{2}n = \frac{30^4 - 20^4}{40^2 - 20^2} \times 22\frac{1}{2} \times \frac{{\cdot}785398 \text{ \&c.}}{2} = \frac{6500}{96} \times 90 \times {\cdot}785398$ &c. $= 4786{\cdot}020058$ cubic inches $= 16{\cdot}971702$ ale gallons $= 20{\cdot}718701$ wine gallons.

PROBLEM XIV.

To find the ſolidity of the ſlice ABC *cut off a paraboloid* ADE *by a plane* FCG *parallel to the axe* DH.

RULE.

Multiply the baſe of the ſlice by the ſquare of the diameter of the baſe of the whole paraboloid, and divide



divide the product by the ſquare of the chord of the baſe of the ſlice; from the quotient ſubtract $\frac{1}{3}$ of the product of the ſaid chord and its diſtance from the center of the baſe of the paraboloid; and the difference multiplied by half the altitude of the ſlice will be the ſolidity required.

That is, $\overline{\frac{FAG \times EA^2}{FG^2} - \frac{FG \times BH}{3}} \times \frac{1}{2} BC =$ the ſlice ABC.

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* DEMONSTRATION.

By corollary 4 prop. 1 ſect. 2, the ſection FCG is a parabola having the ſame parameter with the generating parabola EDA, and therefore $\frac{2}{3} FG \times CB \times B\dot{A}$ is the fluxion of the ſolid: but, putting $a = DH$, $r = HA$, $x = AB$, and $y = BG$, by the property of the parabola, it will be $rr : a :: yy : \frac{ayy}{rr} = CB$; and, by the property of the circle, $r - x = \sqrt{rr - yy}$; hence $B\dot{A} = \dot{x} = \frac{y\dot{y}}{\sqrt{rr - yy}}$; then, by ſubſtituting theſe values, the fluxion of the ſolid will become $\frac{4a}{3rr} \times \frac{y^4 \dot{y}}{\sqrt{rr - yy}}$, whoſe fluent is $a \times$ area $BAG - \frac{ay^3 \sqrt{rr - yy}}{3rr}$ = the ſolid required.

But, putting A for the altitude BC, it will be $yy : A :: rr : a = \frac{Arr}{yy}$, which being ſubſtituted for it in the above found value of the ſolid, gives $A \times \frac{rr}{yy} \times \overline{BAG - \frac{1}{3} y \sqrt{rr - yy}} = \frac{1}{2} BC \times \overline{\frac{EA^2}{FG^2} \times FAG - \frac{1}{3} FG \times BH} =$ the ſlice ACB. Q.E.D.

Co-

EXAMPLE I.

If a caſk, compoſed of two equal paraboloidal fruſtums, having its length = 40, its bung diameter 32, and its head diameter 24 inches, when laid with its axe parallel to the horizon, want ſo much of being full that there are $3\frac{1}{5} = 3{\cdot}2$ inches of the bung diameter left dry; it is required to find how many ale gallons will be neceſſary to fill the caſk.

Here 3·2 is the verſed ſine of the baſe of the ſlice, and $\frac{3{\cdot}2}{32} = {\cdot}1$ = the tabular verſed ſine, to which, in the table, correſponds the area ·04087528, and hence $\cdot 04087528 \times 32^2 = 41{\cdot}85628672$ = the baſe of the ſlice.

COROLLARY I.

The ſlice ABC is equal to a paraboloid of the ſame altitude, and whoſe baſe is equal to the difference between $\frac{2}{3}$ of the rectangular area $HBGI$ and a fourth proportional to the areas BG^2, HA^2, $FAGF$.

COROLLARY II.

From the demonſtration it appears that the ſlice is alſo equal to $a \times BAG - \frac{y^3\sqrt{rr-yy}}{3rr}$.

COROLLARY III.

When B coincides with H, A will be $= a$, and $r = y$; and therefore the rule becomes $a \times HAK$ = half the paraboloid.

COROLLARY IV.

From $a \times HAK$ = half the paraboloid, ſubtract $a \times BAG - \frac{ay^3\sqrt{rr-yy}}{3rr}$ = the ſlice ABC, and there will remain $a \times HBGK + \frac{ay^3\sqrt{rr-yy}}{3rr} = a \times HBGK + \frac{1}{3}yy\sqrt{DL \times LH}$ for the value of the complemental ſlice $DHBC$; CL being drawn parallel to HA.

But, by the property of the circle, $2\sqrt{\overline{32 - 3{\cdot}2} \times 3{\cdot}2} = 2\sqrt{28{\cdot}8 \times 3{\cdot}2} = 19{\cdot}2$ = the chord of the baſe of the ſlice, and $16 - 3{\cdot}2 = 12{\cdot}8$ = its diſtance from the center or middle of the bung diameter. Alſo, by the nature of the parabola, $\overline{16^2 - 12^2} : 20 :: 9{\cdot}6^2 : \frac{9{\cdot}6^2 \times 20}{28 \times 4} = \frac{24^2}{35} = 16\frac{16}{35}$ = the altitude of the ſlice, or half the length of the empty part.

Then, $\overline{\frac{41{\cdot}85628672 \times 32^2}{19{\cdot}2 \times 19{\cdot}2} - \frac{19{\cdot}2 \times 12{\cdot}8}{3}} \times \frac{24^2}{35} =$
$\overline{\frac{41{\cdot}85628672 \times 5^2}{3^2} - 6{\cdot}4 \times 12{\cdot}8} \times \frac{24^2}{35} =$
$\overline{41{\cdot}85628672 \times 5 - {\cdot}4 \times 19{\cdot}2^2} \times \frac{8^2}{7} = 565{\cdot}2611072$ cubic inches $= 2{\cdot}004472$ ale gallons = the content of the empty part required.

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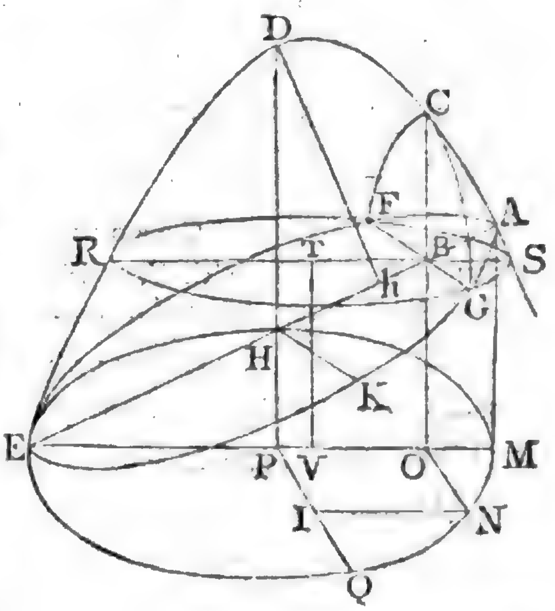
COROLLARY V.



After the ſame manner may be found the ſlice parallel to the axe of an oblique ſegment ADE.

Thus, ſince $\frac{2}{3}FG \times CB \times \dot{B}A \times$ s. ∟ ABC is the fluxion of the ſlice ACB, by putting $a = DH$ the diameter of the double ordinate AE, $t = HA$ the ſemi-tranſverſe axe of the ſection $AFEGA$, $c = HK = \frac{1}{2}EM$ = the ſemi-conjugate, $x = AB$, and $y = BG$; we ſhall have $cc : a :: yy : \frac{ayy}{cc} = BC$; but $cc : tt :: yy : tt - \overline{t - x}|^2$, hence

EXAMPLE II.

If the ſame caſk be placed in the ſame manner, and $6\frac{2}{5}$ inches of the bung diameter be dry; required how many ale gallons the caſk wants of being full.

Here the dry inches being more than $16-12=4$ the difference of the bung and head ſemi-diameters, ſhews that $6\frac{2}{5}-4=2\frac{2}{5}$ inches of the head diameter alſo

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hence $t-x=\frac{t\sqrt{cc-yy}}{c}$, and $\dot{x}=B\dot{A}=\frac{ty\dot{y}}{c\sqrt{cc-yy}}$; alſo s. $\angle ABC$ is $=\frac{c}{t}$; conſequently the above fluxion will become $=\frac{4}{3}y \times \frac{ayy}{cc}\times\frac{ty\dot{y}}{c\sqrt{cc-yy}}\times\frac{c}{t}=\frac{4ay^4\dot{y}}{3cc\sqrt{cc-yy}}$; the fluent of which viz. $\frac{ac}{t}\times$ elliptic area $BAG-\frac{ay^3\sqrt{cc-yy}}{3cc}$ will be the value of the ſlice BCA.

Or, if Db be drawn perpendicular to AE, ſince, by ſimilar triangles, it is $t:c::a:\frac{ac}{t}=Db$, the ſame ſlice BCA will be expreſſed by $Db\times BAG-\frac{ay^3\sqrt{cc-yy}}{3cc}=Db\times BAG-\frac{DH\times FG^3\sqrt{EM^2-FG^2}}{12EM^2}$.

And if CB be produced to meet EM in O, and ON be perpendicular to EM and meet the circle ENM in N; then will the circle ENM be to the ellipſe EGA, as well as the ſegment NMO to the ſegment GAB, as ME to EA, as c to t; and therefore the value of the ſlice BCA will alſo be $a\times MNO-\frac{ay^3\sqrt{cc-yy}}{3cc}=DH\times MNO-\frac{DH\times FG^3\sqrt{EM^2-FG^2}}{12EM^2}$. Where the circular area MNO is the projection of the elliptic area ABG, by lines parallel to the diameter DH upon a plane perpendicular to the ſame.

More

alſo is dry; and therefore the empty part of the caſk will be double the difference of two ſlices the verſed ſines of whoſe baſes are $6\frac{2}{5}$ and $2\frac{2}{5}$, the whole diameters being 32 and 24.

Now the tabular verſed ſines will be $\frac{6{\cdot}4}{32} = {\cdot}2$, and $\frac{2{\cdot}4}{24} = {\cdot}1$, whoſe tabular areas are ·11182380 and ·04087528; hence $32^2 \times {\cdot}1118238 = 114{\cdot}5075712 =$ the baſe of the greater ſlice, and $24^2 \times {\cdot}04087528 = 23{\cdot}54416128 =$ that of the leſs.

Again,

Moreover, ſince $yy : cc :: BC = A : a = HD = \frac{Acc}{yy}$, by ſubſtituting this value of a inſtead of it, the ſolidity of the ſame ſlice will become $\frac{Ac^3}{tyy} \times AGB - \frac{1}{3} Ay\sqrt{c^2 - y^2} = BC \times \overline{\frac{PM^2}{NO^2} \times MNO - \frac{1}{3} NO \times OP}$.

COROLLARY VI.

Hence the ſlice $FGAC$ is equal to a paraboloid whoſe altitude is BC, and its baſe equal to the difference between $\frac{1}{3}$ of the rectangular area $PINO$ and the fourth proportional to the areas NO^2, MP^2, $2MNO$.

COROLLARY VII.

When R coincides with H, the rule becomes barely $DH \times MQP = Db \times HKA$ for the value of $DAH = DEH =$ half the oblique ſegment.

COROLLARY VIII.

By taking the ſlice from the ſemi-ſegment, we obtain $DH \times PONQ + DH \times \frac{y^3\sqrt{cc - yy}}{3cc} = DH \times PONQ + DH \times \frac{NO^3 \times OP}{3PM^2}$ for the value of the complemental ſlice $DCBH$.

COROLLARY IX.

And by adding this laſt to the ſemi ſegment, there reſults $DH \times ENO + DH \times \frac{NO^3 \times OP}{3PM^2} = \frac{BC \times PM^2}{ON^2} \times ENO + BC \times \frac{1}{3} NO \times OP$ for the value of the ſlice $EDCBE$ greater than the ſemi-ſegment.

Co-

Again, $16-6{\cdot}4=9{\cdot}6=$ the diſtance of the chord of the baſe of the greater ſlice from the middle of the bung diameter, and $12-2{\cdot}4=9{\cdot}6=$ the diſtance of that of the leſs from the middle of the head diameter; and hence the former chord will be $=2\sqrt{16^2-9{\cdot}6^2}=2\sqrt{25{\cdot}6\times6{\cdot}4}=25{\cdot}6$, and the latter $=2\sqrt{12^2-9{\cdot}6^2}=2\sqrt{21{\cdot}6\times2{\cdot}4}=14{\cdot}4$.

Moreover, by the nature of the parabola, as

$$16^2-12^2 \text{ or } 28\times4:20::\left\{\begin{array}{l}12{\cdot}8^2:\dfrac{20\times3{\cdot}2^2}{7}=\dfrac{32^2}{35}=29\tfrac{9}{35}\\[2ex] 7{\cdot}2^2:\dfrac{20\times1{\cdot}8^2}{7}=\dfrac{18^2}{35}=9\tfrac{9}{35}\end{array}\right\}$$

= the altitudes of the ſlices.

Conſequently, $\overline{\dfrac{114{\cdot}5075712\times32^2}{25{\cdot}6^2}-\dfrac{25{\cdot}6\times9{\cdot}6}{3}}\times\dfrac{32^2}{35}=$ $\overline{\dfrac{114{\cdot}5075712\times5^2}{4^2}-25{\cdot}6\times3{\cdot}2}\times\dfrac{32^2}{35}=$ $\overline{114{\cdot}5075712\times5-{\cdot}4\times25{\cdot}6^2}\times\dfrac{8^2}{7}=2837{\cdot}886683=$ the double of the greater ſlice.

COROLLARY X.

If $RGSFR$ be a circle perpendicular to BC, and if from its center T be demitted the perpendicular TV; then, by taking the oblique ſlice ABC from the right ſlice SBC, we ſhall obtain $BC\times\dfrac{RS^2\times BSG-EM^2\times OMN+FG^3\times\frac{1}{6}PV}{FG^2}$ for the value of the ungula or cuneus $AFGSA$ contained between the two ſections $RFSGR$, $EFAGE$. And the ſame will hold though both the ſections be oblique

COROLLARY XI.

And, in the ſame manner, $BC\times\dfrac{EM^2\times OEN-RS^2\times BRG+FG^3\times\frac{1}{6}PV}{FG^2}$ will be the value of the oppoſite ungula or cuneus $EGFRE$.

And $\overline{\frac{23{\cdot}54416128 \times 24^2}{14{\cdot}4^2} - \frac{14{\cdot}4 \times 9{\cdot}6}{3}} \times \frac{18^2}{35} =$ $\overline{\frac{23{\cdot}54416128 \times 5^2}{3^2} - \frac{14{\cdot}4 \times 9{\cdot}6}{3}} \times \frac{18^2}{35} =$ $\overline{23{\cdot}54416128 \times 5 - {\cdot}4 \times 14{\cdot}4^2} \times \frac{6^2}{7} = 178{\cdot}852147 =$ the double of the leſs.

Wherefore $2837{\cdot}886683 - 178{\cdot}852147 = 2659{\cdot}034536$ inches $= 9{\cdot}4292$ ale gallons $=$ the quantity required.

PROBLEM XV.

To find the ſolidity of a parabolic ſpindle.

* RULE.

Multiply the area of the greateſt circle, or middle ſection, by the length, and $\frac{8}{15}$ of the product will be the content.

That

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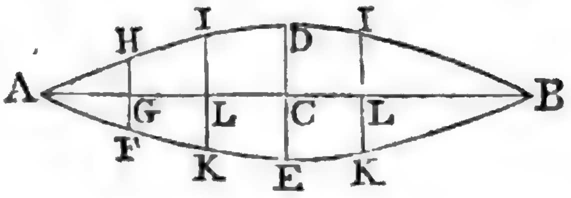
* DEMONSTRATION.

Putting $a = DC$, $b = CA$, $c = 3{\cdot}14159$ &c. $x = AG$, and $y = GH$; we ſhall have, by the property of the parabola, as $AC^2 : CD :: AG \times GB : GH = a \times \frac{2bx - xx}{bb} = \frac{2bx - xx}{p} = y$, putting $p = \frac{bb}{a} =$ the parameter of DC; and hence the fluxion of the ſolid $cyy\dot{x}$ will be $\frac{c\dot{x}}{pp} \times \overline{2bx - xx}|^2$; the fluent of which $cx^3 \times \frac{\frac{4}{3}bb - bx + \frac{1}{5}xx}{pp}$ will be a general expreſſion for the ſegment AFH; which, when $x = b$, gives $\frac{8cb^5}{15pp} = \frac{8}{15}caab$ for the value of the ſemi-ſpindle ADE. Q.E.D.

Co-



That is, $\frac{8}{15}n \times DE^2 \times AB =$ the whole ſolid $ADBEA$; AB being the length or axe, DE the greateſt diameter or double the abſciſſa of the generating parabola ADB, and $n =$ ·785398 &c.

EXAMPLE.

Required the ſolidity of a parabolic ſpindle whoſe length is 80 and greateſt diameter 32.

By the rule, ·785398 &c. $\times 32^2 \times 80 \times \frac{8}{15} =$ ·785398 &c. $\times 43690\frac{2}{3} = 34314{\cdot}5693576 =$ the content required.

PRO-

COROLLARY I.

A parabolic ſemi-ſpindle is to a cylinder of the ſame baſe and altitude as 8 to 15, and to a paraboloid of the ſame baſe and altitude as 8 to $7\frac{1}{2}$ or as 16 to 15.

COROLLARY II.

It appears from the demonſtration that the ſegment AHF is expreſſed by $cx^3 \times \frac{\frac{4}{3}bb - bx + \frac{1}{5}xx}{pp} = c \times AG^3 \times DC^2 \times \frac{\frac{4}{3}AC^2 - CA \times AG + \frac{1}{5}GA^2}{AC^4}$.

COROLLARY III.

If from the value of the ſemi-ſpindle ADE be taken that of the ſegment AFH, there will remain $\frac{8cb^5}{15pp} - cx^3 \times \frac{\frac{4}{3}bb - bx + \frac{1}{5}xx}{pp} = \frac{cz}{pp}\left(\text{or } \frac{ca^2z}{b^4}\right) \times \overline{b^4 - \frac{2}{3}b^2z^2 + \frac{1}{5}z^4}$ for the value of the fruſtum $EFHD$, putting z inſtead of $b - x = CG$.——And if inſtead of z,

in

PROBLEM XVI.

To find the content of the ſegment AFH *of a parabolic ſpindle.*

RULE.

To $\frac{4}{3}$ of the ſquare of the length of the ſemi-ſpindle, add $\frac{1}{5}$ of the ſquare of the altitude of the ſegment, from the ſum take the product of the ſaid length and altitude, and call the difference P.

Mul-

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in the two laſt terms of this expreſſion, be ſubſtituted its value $b\sqrt{\frac{a-y}{a}}$, the value of the ſaid fruſtum will be denoted by $cz \times \frac{8aa + 4ay + 3yy}{15} = c \times GC \times \frac{8DE^2 + 4DE \times FH + 3FH^2}{60}$.

COROLLARY IV.

If to, or from, the fruſtum $DEFH = cA \times \frac{8D^2 + 4Dd + 3d^2}{60}$, be added, or ſubtracted, the fruſtum $DIKE = ca \times \frac{8D^2 + 4D\delta + 3\delta^2}{60}$, there will reſult $c \times \frac{8D^2 \times \overline{A \pm a} + dA \times \overline{4D + 3d} \pm \delta a \times \overline{4D + 3\delta}}{60}$ for the value of the fruſtum $HIKF$, neither of whoſe ends paſs through the center of the ſpindle, the upper or under ſigns being uſed according as the ſaid center is within or without the fruſtum, and wherein D repreſents the diameter through the center of the ſpindle, d the diameter of the leſs end of the fruſtum, and δ that of the greater; alſo A the diſtance of d from the center of the ſpindle, and a that of δ from the ſame. Or in a caſk of this form, ſtanding upon one end, D will be the bung diameter, d the head diameter, $2A$ the length of the caſk, $A \pm a = w =$ the wet inches of $2A$, and δ the diameter at the ſurface of the liquor, the value of which diameter is as determined in the following corollary.

COROLLARY V.

Since, by the property of the parabola, $A^2 : a^2 :: D-d : D-\delta$, the value of δ will be $\frac{DA^2 - a^2 \times \overline{D-d}}{AA}$.

Multiply the cube of the altitude of the ſegment by the ſquare of the greateſt diameter of the ſpindle, divide the product by the fourth power of the length of the ſemi-ſpindle, and call the quotient Q.

Then the product of P and Q multiplied by ·785398 &c. will give the content of the ſegment.

That is, $\frac{\frac{4}{3}AC^2 - CA \times AG + \frac{1}{5}GA^2}{AC^4} \times AG^3 \times DE^2 \times$ ·785398 &c. = the ſegment AFH, by corollary 2 of the laſt problem.

EXAMPLE.

If the diameter of the baſe of the ſegment of a parabolic ſpindle be 24, and its altitude 20; what will be the content, ſuppoſing the length of the whole ſpindle to be 80.

Here $AC = 40$, $AG = 20$, and $GH = 12$; then $GC = 40 - 20 = 20$, and, by the parabola, $AC^2 - CG : GH :: AC^2 : CD = \frac{AC^2 \times GH}{AC^2 - CG^2} = \frac{40^2 \times 12}{60 \times 20} = 4 \times 4 = 16$; the double of which gives $DE = 32$.

Then, by the rule, $\overline{\frac{4 \times 40^2}{3} - 40 \times 20 + \frac{20^2}{5}} \times \frac{20^3 \times 32^2}{40^4} \times$ ·785398 &c. $= \overline{\frac{4}{3} - \frac{1}{3} + \frac{1}{20}} \times 5 \times 32^2 \times$ ·785398 &c. $= \frac{80 - 30 + 3}{3} \times 16^2 \times$ ·785398 &c. $= 4522\frac{2}{3} \times$ ·785398 &c. = 3552·094093659 = the content required.

PROBLEM XVII.

To find the content of the fruſtum of a parabolic ſpindle, one of the ends of the fruſtum paſſing through the center of the ſpindle.

RULE.

Add into one ſum 8 times the ſquare of the diameter of the greater end, 3 times the ſquare of that of the leſs, and 4 times the product of the diameters; multiply the ſum by the length, and the product multiplied by $\frac{1}{15}$ of ·785398 &c. will be the content.

8 That is, $\frac{\cancel{3}DE^2 + 4DE \times FH + 3FH^2}{15} \times CG \times {}$·785398 &c. = the fruſtum *DEFH*, by corollary 3 to prob. 15.

EXAMPLE.

Required the content of a caſk in form of the middle zone of a parabolic ſpindle, the bung diameter being 32 inches, the head diameter 24, and length 40 inches.

The caſk is evidently two equal fruſtums, whoſe greater diameters are 32 and leaſt 24, and their lengths each 20; wherefore $\frac{8 \times 32^2 + 4 \times 32 \times 24 + 3 \times 24^2}{15} \times 40 \times {}$·785398 &c. $= 8^3 \times \frac{8 \times 4^2 + 3 \times 4^2 + 3 \times 3^2}{3} \times {}$·785398 &c. $= \frac{11 \times 16 + 27}{3} \times 8^3 \times {}$·785398 &c. $= 34645\frac{1}{3} \times {}$·785398 &c. = 27210·3811703 inches = 96·49071337 ale gallons = the content required.

PROBLEM XVIII.

To find the content of the fruſtum of a parabolic ſpindle, neither of whoſe ends paſs through the center of the ſpindle.

RULE.

Multiply the diameter of each end by its diſtance from the diameter in the middle of the ſpindle, and multiply each product by the ſum of 3 times the ſaid diameter of the end and 4 times the ſaid middle diameter;

diameter ;to or from the product belonging to the leſs diameter add or ſubtract that belonging to the greater, according as the center of the ſpindle is within or without the fruſtum; to the ſum or difference add 8 times the product ariſing from the multiplication of the length of the fruſtum by the ſquare of the diameter in the middle of the ſpindle; and this ſum multiplied by $\frac{1}{15}$ of ·785398 &c. will produce the content of the fruſtum.

That is,

$$\frac{8DE^2 \times GL + FH \times GC \times \overline{4DE + 3FH} \pm IK \times LC \times \overline{4DE + 3IK}}{15}$$

$\times \cdot785398$ &c. = the fruſtum *FHIK*, uſing the upper or under ſign according as GL is greater or leſs than GC; as is proved in corollary 4 to prob. 15.

Note, The value of the greater diameter IK is $\frac{DE \times GC^2 - LC^2 \times \overline{DE - FH}}{GC^2}$, as in corollary 5 to the ſame problem.

EXAMPLE I.

If the liquor in the caſk, in the example to the laſt problem, when ſtanding upon its end with its axe perpendicular to the horizon, riſe to the height of 30 inches; what quantity of liquor will there be.

Here, by the note, $IK = \frac{32 \times 20^2 - 10^2 \times \overline{32 - 24}}{20 \times 20} = 30$.

Then, by the rule,

$$\frac{8 \times 32^2 \times 30 + 24 \times 20 \times \overline{4 \times 32 + 3 \times 24} + 30 \times 10 \times \overline{4 \times 32 + 3 \times 30}}{15} \times$$

$\cdot785398$ &c. $= \overline{8 \times 16^2 + 4 \times 128 + 72 + 5 \times 64 + 45} \times 8 \times \cdot785398$ &c. $= 27144 \times \cdot785398$ &c. $=$ 21318·8477473 cubic inches = 75·59875088 ale gallons = the quantity required.

Ex-

EXAMPLE II.

If in the ſame caſk, placed as before, the liquor riſe but to the height of 10 inches; how much will be in it.

Here all the dimenſions are the ſame quantities as in the laſt example, excepting GL which here is only 10 inſtead of 30, and the caſk is leſs than half full.

Wherefore, by the rule,

$$\frac{8\times32^2\times10+24\times20\times\overline{4\times32+3\times24}-30\times10\times\overline{4\times32+3\times30}}{15}$$

$$\times\cdot785398\text{ \&c.}=\frac{8\times16^2+12\times\overline{128+72}-15\times\overline{64+45}}{3}\times8\times$$

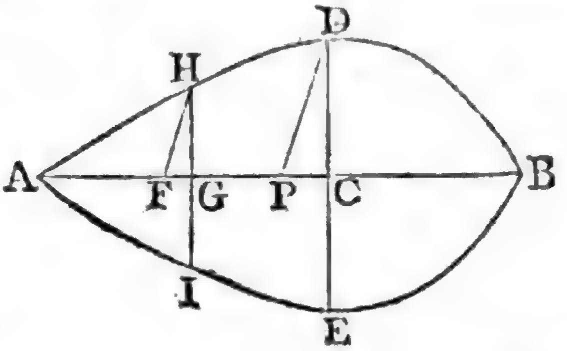
$\cdot785398$ &c. $= 7501\frac{1}{3}\times\cdot785398$ &c. $= 5891\cdot533423$ cubic inches $= 20\cdot89196249$ ale gallons = the quantity required.

PROBLEM XIX.

To find the content of the univerſal parabolic ſpindle, or ſolid generated by the revolution of a parabolic ſegment round its baſe, being an ordinate to any diameter.

RULE.

Multiply the area of the greateſt ſection by the length, and $\frac{8}{15}$ of the product will be the content; as in the common ſpindle.



That is, $DE^2\times AB\times\frac{8}{15}n=$ the ſolid $ADBEA$, generated from the revolution of the parabolic ſegment ADB about its baſe AB which is a double a double ordinate to the diameter DP, and DCE being perpendicular to AB.*

E X A M P L E.

If an oblique parabola, whoſe baſe is inclined to its diameter in an angle of 30 degrees, be turned about its baſe; required the value of the ſolid generated by its revolution; the baſe of the parabola,

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* DEMONSTRATION.

Let GH be parallel to CD, and HF parallel to DP; and put $n = DP$, $\frac{1}{2}b = PA = \frac{1}{2}AB$, $\frac{1}{2}c = CD = \frac{1}{2}DE$, $z = AF$, and $p = 4n = 3{\cdot}14159$ &c. Then, by the property of the parabola, $AP^2 : AF \times FB :: PD : FH ::$ (by ſim. △s.) $CD : GH = \frac{CD \times BF \times FA}{AP^2} = 2c \times \frac{bz - zz}{bb} :: CP = \sqrt{PD^2 - DC^2} = d :$ $GF = \frac{GH \times PC}{CD} = 4d \times \frac{bz - zz}{bb}$: hence $AG = GF + FA = z \times \frac{bb + 4bd - 4dz}{bb}$, and the fluxion $p \times HG^2 \times \dot{GA}$ of the ſolid will be $p\dot{z} \times \frac{bb + 4bd - 8dz}{bb} \times \frac{4cc}{b^4} \times \overline{bz - zz}|^2 = 4pc^2z^2\dot{z} \times$ $\frac{b^3 \times \overline{b + 4d} - 2b^2z \times \overline{b + 8d} + bz^2 \times \overline{b + 20d} - 8dz^3}{b^6}$; the fluent of which, viz. $4pc^2z^3 \times$ $\frac{\frac{1}{3}b^3 \times \overline{b + 4d} - \frac{1}{2}b^2z \times \overline{b + 8d} + \frac{1}{5}bz^2 \times \overline{b + 20d} - \frac{4}{3}dz^3}{b^6}$ will be a general expreſſion for the value of the ſegment; which, when z becomes $= b$, will be $\frac{2}{15}pccb = \frac{8}{15}nbcc$ for the content of the whole ſolid $ADBEA$. Q.E.D.

COROLLARY.

If any parabolas ADB, ACB, [fig. at corol. 1 to prob. 5] having the ſame baſe AB and between the ſame parallels AB, CD, be turned round their common baſe; they will generate equal ſpindles.

For the lengths and greateſt diameters will be the ſame in all of them.

or length of the ſpindle, being 80, and the diameter from the vertex to the baſe 32.

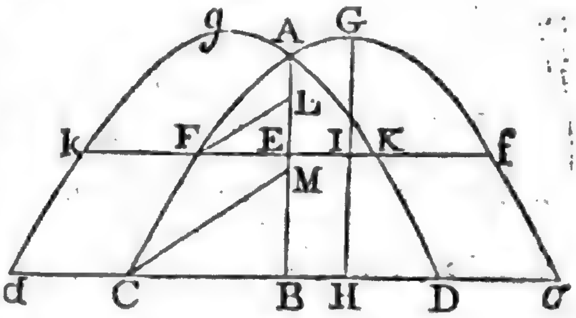
Here $AB=80$, $PD=32$, and the angle $DPC=30°$; then, as radius $=1:\frac{1}{2}=$ ſine $\angle P$ or $30°::PD=32:DC=16=$ the radius of the greateſt circle of the ſpindle.

Wherefore $32^2\times80\times\frac{8}{15}\times{}^{\cdot}785398$ &c. $={}^{\cdot}785398$ &c. $\times43690\frac{2}{3}=34314{}^{\cdot}5693576$ as in the example to prob. 15.

PROBLEM XX.

To find the value of an univerſal paraboloid, or body generated by the revolution of a parabola round any diameter.

INVESTIGATION.



Suppoſe AB to be the diameter about which the parabola CAc revolves, CAD a vertical ſection of the ſolid generated by AC, and $dgAGc$ that of the ſolid generated by cGA; alſo let GIH be perpendicular, and $FEIF$ parallel to Cc. Then, by the property of the parabola, $CH^2=bb:HG=a::CH^2-IF^2=CH^2-\overline{FE\mp EI}|^2=bb-\overline{y\mp d}|^2:BE=x::-2\dot{y}\times\overline{y\mp d}:\dot{x}=-2a\dot{y}\times\frac{y\mp d}{bb}$; hence the fluxion of the ſolid $2cyy\dot{x}$ will become $-2acy^2\dot{y}\times\frac{y\mp d}{bb}$; and the correct fluent gives $ac\times\frac{\overline{b\pm d}|^3\times\overline{3b\mp d}-y^3\times\overline{3y\mp4d}}{6bb}=$

$ac \times \frac{\overline{b \pm d}|^3 \times \overline{3b \mp d} - \overline{b\sqrt{\frac{a-x}{a}} \pm d}|^3 \times \overline{3b\sqrt{\frac{a-x}{a}} \mp d}}{6bb}$ for the value of the ſolid required, in which the upper ſigns reſpect the ſolid *dkfc* generated by *cf*, and the under the ſolid *DKFC* generated by *CF*.

COROLLARY I.

When E arrives at A, then $y = 0$, and the expreſſion becomes $ac \times \overline{b \pm d}|^3 \times \frac{3b \mp d}{6bb} = c \times HG \times CB^3 \times \frac{cB + 2CH}{6CH} = \Big($by ſubſtituting for HG its value $\frac{CH^2 \times BA}{CB \times Bc}\Big)$ $c \times CB^2 \times BA \times \frac{Bc + 2CH}{6Bc}$ for the ſolid CAD.

And if CM be an ordinate to the diameter AB, we ſhall have as $cB : BC :: BA : AM = \frac{AB \times BC}{Bc}$; hence $BA \times \frac{Bc + 2CH}{6Bc}$ will be $= \frac{2BA + AM}{6}$, and conſequently the above value of the ſolid CAD becomes $c \times CB^2 \times \frac{2BA + AM}{6}$

COROLLARY II.

When B coincides with H, d will be $= 0$, and then the rule becomes $ac \times \frac{b^4 - y^4}{2bb} = c \times \frac{1}{2}HI \times \overline{bb + yy}$ for the fruſtum $CFfc$ generated about HI, as in problem 12. Or $c \times CH^2 \times \frac{1}{2}HG$ for the whole paraboloid, as in prob. 11.

The INVESTIGATION *otherwiſe.*

Suppoſing FL to be an ordinate to the diameter AB whoſe parameter is p, and putting m and n for the ſine and coſine of the angle L to the radius 1; we ſhall have $FL = \sqrt{p} \times LA = \sqrt{px}$, where x is now $= LA$; then $FE = m\sqrt{px}$, and $EL = n\sqrt{px}$; hence $EA = AL + LE = x + n\sqrt{px}$. Then the fluxion of the ſolid $c \times FE^2 \times EA = cm^2px \times \overline{\dot{x} + \frac{1}{2}n\dot{x}\sqrt{\frac{p}{x}}}$, whoſe fluent gives $cm^2px \times \overline{\frac{1}{2}x + \frac{1}{3}n\sqrt{px}} = cyy \times \frac{3x + 2n\sqrt{px}}{6} = c \times FE^2 \times \frac{2EA + AL}{6}$ for the value of the ſolid AFK generated by the revolution of AF. And $c \times CB^2 \times \frac{2BA + AM}{6}$ for the ſolid CAD as above.

SCHOLIUM.

It is evident that the ſurface of this ſolid might be found by the method by which we determined that of an elliptic ſpindle; and the ſurface of the parabolic ſpindle alſo might eaſily be determined if there appeared an occaſion for it, but not by the ſame method.

SECTION V.

Of hyperbolic Lines, Areas, Surfaces, and Solidities.

DEFINITIONS.

TO what hath been ſaid of the hyperbola among the conic ſections in general, may be added the following obſervations.

If AVB touch the hyperbola in the vertex V, and be equal to the conjugate axe, viz. AV and VB each equal to the half of the conjugate axe, and if from the

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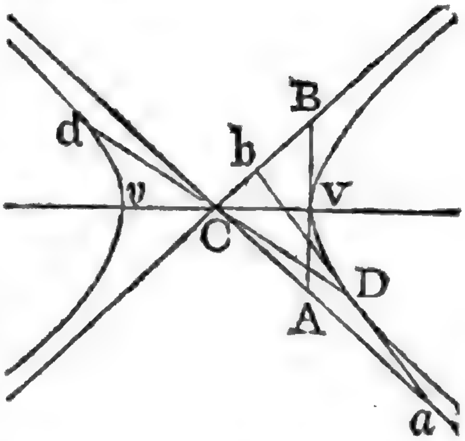


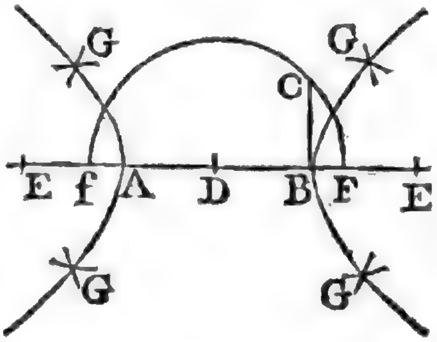
the center *C* through the extremities *A*, *B*, right lines *CA*, *CB*, be drawn, thoſe lines will be what are called aſymptotes to the hyperbola; that is, they are lines which continually approach to the curve.

And like as the conjugate axe is a tangent to the curve at the vertex of the tranſverſe and bounded by the aſymptotes, ſo if to the extremity *D* of any diameter *Dd* be drawn a tangent *aDb* and terminated by the aſymptotes, then *ab* will be the conjugate to *Dd*.

When the ſymptotes form a right angle at the center, the hyperbola is ſaid to be right-angled; or, alſo equilateral, becauſe that then its axes are equal to each other.

PROBLEM I.

To conſtruct an hyperbola, having given the tranſverſe and conjugate axes AB *and* 2*BC*.



The ſemi-conjugate *BC* being erected perpendicular to *AB*, and *D* being the middle of *AB* or the center of the hyperbola; with the center *D* and radius *DC*, deſcribe an arc meeting with *AB*, both ways produced, in *F* and *f* the two focuſes.

Then, aſſuming ſeveral points E in the tranſverſe, produced, with the radiuſes AE, BE, and centers f, F, deſcribe arcs interſecting in the ſeveral points G; through all which draw the curve.

PROBLEM II.

In an hyperbola, to find any two conjugate diameters, an ordinate to one of them, and its abſciſſa, one from another; viz. having any three of them given, to find the fourth.

CASE I.

To find the ordinate.

*RULE.

As any diameter:
Is to its conjugate::
So is the mean proportional between the two abſciſſas:
To the ordinate.

That is, as $d : c :: \sqrt{\overline{d+x} \times x} : \frac{c}{d}\sqrt{\overline{d+x} \times x} = y$; where d denotes the diameter, c its conjugate, y an ordinate to the diameter, and x its abſciſſa, or its diſtance from the vertex of the diameter, meaſured upon it.

Note. In the hyperbola, the leſs abſciſſa added to the diameter, gives the greater abſciſſa.

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* The values of the ſeveral quantities in the four caſes of this problem are eaſily found from the general property of the curve, viz. $dd : cc :: x \times \overline{d+x} : yy$. The demonſtration of which together with that of the preceding problem belongs to conics.

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EXAMPLE.

If the diameter be 24, its conjugate 21, and the leſs abſciſſa 8; what is the ordinate.

By the rule, $\frac{c\sqrt{\overline{d+x}\cdot x}}{d} = \frac{21\sqrt{\overline{24+8}\cdot 8}}{24} = \frac{7\times 16}{8} = 14$ = the ordinate required.

CASE II.

To find the abſciſſas.

RULE.

As the conjugate:
Is to the diameter::
So is the root of the ſum of the ſquares of the ordinate and ſemi-conjugate:
To the diſtance between the bottom of the ordinate and the center.

Then, to and from this diſtance add and ſubtract the ſemi-diameter, and the ſum and difference will give the two abſciſſas.

That is, $\frac{d\sqrt{\frac{1}{4}cc+yy}}{c} \pm \frac{1}{2}d = x$ the greater or leſs abſciſſa according as the upper or under ſign is uſed.

EXAMPLE.

The diameter and its conjugate being 24 and 21, required the two abſciſſas to the ordinate 14.

Here $\frac{d\sqrt{\frac{1}{4}cc+yy}}{c} \pm \frac{1}{2}d = \frac{24\sqrt{\left.\frac{21}{2}\right|^2 + 14^2}}{21} \pm 12 = \frac{4\sqrt{21^2+28^2}}{7} \pm 12 = 4\sqrt{3^2+4^2} \pm 12 = 20 \pm 12 =$ 32 and 8 the two abſciſſas required.

CASE III.

To find the conjugate.

RULE.

As the mean proportional between the abſciſſas:
Is to the ordinate ::
So is the diameter:
To its conjugate.
That is, $\sqrt{\overline{d+x} \times x} : y :: d : \frac{dy}{\sqrt{\overline{d+x} \times x}} = c$.

EXAMPLE.

What is the conjugate to the diameter 24, the leſs abſciſſa being 8, and its ordinate 14.

Here $\frac{dy}{\sqrt{\overline{d+x} \times x}} = \frac{24 \times 14}{\sqrt{32 \times 8}} = \frac{24 \times 14}{16} = 21$ = the conjugate required.

CASE IV.

To find the diameter.

RULE.

To or from the root of the ſum of the ſquares of the ordinate and ſemi-conjugate, add or ſubtract the ſemi-conjugate, according as the leſs or greater abſciſſa is uſed: Then, as the ſquare of the ordinate is to the product of the abſciſſa and conjugate, ſo will the ſum or difference, above found, be to the diameter.

That is, $cx \times \frac{\sqrt{\frac{1}{4}cc + yy} \pm \frac{1}{2}c}{yy} = d$.

Ex-

EXAMPLE.

The leſs abſciſſa being 8, and its ordinate 14, required the diameter, ſuppoſing the conjugate to be 21.

Here $cx \times \frac{\sqrt{\frac{1}{4}cc+yy}+\frac{1}{2}c}{yy} = 21 \times 8 \times \frac{\sqrt{\overline{\frac{21}{2}}\big|^{2} + 14^{2}} + \frac{21}{2}}{14 \times 14}$ $= 3 \times \frac{\sqrt{21^{2}+28^{2}}+21}{7} = 3 \times \sqrt{3^{2}+4^{2}}+3 = 3 \times \overline{5+3} =$ $24 =$ the diameter required.

PROBLEM III.

Having given any two abſciſſas X, x, *and their ordinates* Y, y, *to find the diameter* d *to which they belong.*

* RULE.

Multiply each abſciſſa by the ſquare of the ordinate belonging to the other; multiply, alſo, the ſquare of each abſciſſa by the ſquare of the other's ordinate; then divide the difference of the latter products by the difference of the former, and the quotient will be the diameter to which the ordinates belong.

That is, $\frac{XXyy \backsim xxYY}{xYY \backsim Xyy} = d.$

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* DEMONSTRATION.

By the nature of the hyperbola, $dX+XX : dx+xx :: YY : yy$, or $dXyy+XXyy = dxYY+xxYY$; hence $d = \frac{XXyy-xxYY}{xYY-Xyy}$. *Q.E.D.*

EXAMPLE.

If two abſciſſas be 1 and 8, and their two ordinates $4\frac{3}{8}$ and 14; what is the diameter to which they belong.

Here $\frac{XXyy - xxYY}{xYY - Xyy} = \frac{8^2 \times 8 \times 4\frac{3}{8} \times 4\frac{3}{8} - 1 \times 1 \times 14 \times 14}{1 \times 14 \times 14 - 8 \times 4\frac{3}{8} \times 4\frac{3}{8}} = \frac{35 \times 35 - 14 \times 14}{14 \times 14 - 35 \times 4\frac{3}{8}} = \frac{5 \times 5 - 2 \times 2}{2 \times 2 - 5 \times \frac{5}{8}} = \frac{21 \times 8}{7} = 24 =$ the diameter required.

PROBLEM IV.

Given any three equidiſtant ordinates a, b, c, *with their common diſtance or common difference* d *of their abſciſſas, to find the diameter* D *to which they belong, together with its conjugate* C.

*RULE.

From the ſum of the ſquares of the extreme ordinates take the double of that of the middle one, and call the difference *A*.

From

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* DEMONSTRATION.

If x be the abſciſſa to the leaſt ordinate a, by the nature of the hyperbola we ſhall have theſe three equations,

$$aa = \frac{CC}{DD} \times \overline{D+x} \times x = \frac{CC}{DD} \times \overline{Dx+xx},$$

$$bb = \frac{CC}{DD} \times \overline{D+x+d} \times \overline{x+d} = \frac{CC}{DD} \times \overline{Dx+xx+2dx+Dd+dd},$$

$$cc = \frac{CC}{DD} \times \overline{D+x+2d} \times \overline{x+2d} = \frac{CC}{DD} \times \overline{Dx+xx+4dx+2Dd+4dd}.$$

And by taking the double of the ſecond from the ſum of the firſt and third, we obtain this equation, $aa - 2bb + cc = \frac{CC}{DD} \times 2dd$; and hence $\frac{CC}{DD} = \frac{aa - 2bb + cc}{2dd}$.

Again,

From the ſquare of the difference between A and the double of the ſaid ſquare of the middle ordinate take 4 times the ſquare of the product of the extreme ordinates, and call the ſquare root of the remainder B.

Then, as A is to B, ſo will the given common diſtance be to the diameter to which the ordinates belong.

That is, $\frac{d\sqrt{\overline{-aa+4bb-cc}|^2-4aacc}}{aa-2bb+cc}=D$.

And as the common diſtance of the ordinates is to the root of $\frac{1}{2}A$, ſo will the diameter D be to its conjugate C.

That is, $C=\frac{D\sqrt{\frac{1}{2}A}}{d}$.

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Again, taking the firſt equation from the ſecond, leaves $bb-aa$ $=\frac{CC}{DD}\times\overline{2dx+Dd+dd}$; hence $x=\frac{DD}{CC}\times\frac{bb-aa}{2d}-\frac{D+d}{2}=$ $d\times\frac{bb-aa}{aa-2bb+cc}-\frac{D+d}{2}=d\times\frac{2bb-\frac{1}{2}aa-\frac{1}{2}cc}{aa-2bb+cc}-\frac{1}{2}D$.

Conſequently $aa=\frac{CC}{DD}\times\overline{D+x}\times x=\frac{aa-2bb+cc}{2dd}\times$ $\overline{d\times\frac{2bb-\frac{1}{2}aa-\frac{1}{2}cc}{aa-2bb+cc}}\Big|^2-\frac{1}{4}D^2$; and hence $\frac{d\sqrt{\overline{-aa+4bb-cc}|^2-4aacc}}{aa-2bb+cc}$ $=\frac{dB}{A}=$ the diameter D.

Alſo from the equation $\frac{CC}{DD}=\frac{aa-2bb+cc}{2dd}$, is deduced $\frac{D}{d}\sqrt{\frac{aa-2bb+cc}{2}}=\frac{D\sqrt{\frac{1}{2}A}}{d}=$ the conjugate C. Q.E.D.

COROLLARY.

Hence the ſum of the ſquares of the extreme ordinates is greater than double the ſquare of the middle one.

EXAMPLE.

Required the diameter and its conjugate to which belong the three ordinates $4\frac{3}{8}$, 14, and $\frac{21}{8}\sqrt{65}$, their common diſtance being 7.

Here $\frac{d\sqrt{\overline{4bb-aa-cc}\big|^2-4aacc}}{aa-2bb+cc}=$

$$\frac{7\sqrt{\overline{28^2-\frac{35^2}{8^2}-\frac{65\times21^2}{8^2}}\Big|^2-\frac{4\times65\times35^2\times21^2}{8^4}}}{\frac{35^2}{8^2}-2\times14^2+\frac{65\times21^2}{8^2}}=$$

$\frac{7\sqrt{\overline{32^2-5^2-65\times3^2}\big|^2-65\times10^2\times3^2}}{5^2-2\times16^2+65\times3^2}=\frac{7\times336}{98}=24=$ the diameter D.

And $\frac{24}{7}\sqrt{\frac{35^2}{2\times8^2}-14^2+\frac{65\times21^2}{2\times8^2}}=8\sqrt{\frac{5^2}{2}-16^2+\frac{65\times3^2}{2}}$ $=\frac{3}{2}\sqrt{50-32^2+9\times130}=\frac{3}{2}\sqrt{196}=\frac{3}{2}\times14=21=$ the conjugate C.

PROBLEM V.

To find the length of any arc of an hyperbola, the arc beginning at the vertex.

*RULE I.

Put $a=$ the ſemi-tranſverſe axe, $c=$ the ſemi-conjugate, $q=\frac{cc+aa}{c^4}$, $y=$ an ordinate to the axe drawn from the end of the arc required, $A =$ the hyperbolic logarithm of $\frac{y+\sqrt{cc+yy}}{c} = 2{\cdot}302585093$ $\times$ common logarithm of $\frac{y+\sqrt{cc+yy}}{c}$, $B = \frac{y\sqrt{cc+yy}-ccA}{2}$, $C = \frac{y^3\sqrt{cc+yy}-3ccB}{4}$, $D = \frac{y^5\sqrt{cc+yy}-5ccC}{6}$, $E = \frac{y^7\sqrt{cc+yy}-7ccD}{8}$, &c.

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* DEMONSTRATION.

Putting $x=$ the abſciſſa of the ordinate y, we ſhall obtain, from the nature of the curve, $x=\frac{a\sqrt{cc+yy}}{c}-a$, and therefore $\dot{x}=$

$a y\dot{y}$

Then will the length of the arc be expreſſed by c drawn into the ſeries $A + \frac{q}{2}B - \frac{q^2}{2.4}C + \frac{3q^3}{2.4.6}D - \frac{3.5q^4}{2.4.6.8}E$ &c.

EXAMPLE.

Required the length of the curve correſponding to the ordinate 10, the tranſverſe and conjugate axes being 80 and 60.

Here $a = 40$, $c = 30$, and $y = 10$; hence $q = \frac{cc+aa}{c^4} = 2500$

$\frac{ay\dot{y}}{c\sqrt{cc+yy}}$; hence $\dot{z} = \sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}} = \dot{y}\sqrt{\frac{cc+\frac{cc+aa}{cc}yy}{cc+yy}} = \frac{c\dot{y}}{\sqrt{cc+yy}}\sqrt{1+\frac{cc+aa}{c^4}yy} = \frac{c\dot{y}}{\sqrt{cc+yy}}\sqrt{1+qyy} = \frac{c\dot{y}}{\sqrt{cc+yy}}$ $\times : 1 + \frac{q}{2}y^2 - \frac{q^2}{2.4}y^4 + \frac{3q^3}{2.4.6}y^6 - \frac{3.5q^4}{2.4.6.8}y^8$ &c.

But the fluent of $\frac{\dot{y}}{\sqrt{cc+yy}}$ is the hyp. log. of $\frac{y+\sqrt{cc+yy}}{c} = A$, that of $\frac{y^2\dot{y}}{\sqrt{cc+yy}}$ is $\frac{y\sqrt{cc+yy}-ccA}{2} = B$, that of $\frac{y^4\dot{y}}{\sqrt{cc+yy}}$ is $\frac{y^3\sqrt{cc+yy}-3ccB}{4} = C$, that of $\frac{y^6\dot{y}}{\sqrt{cc+yy}}$ is $\frac{y^5\sqrt{cc+yy}-5ccC}{6} = D$, &c.

And therefore z is $= c \times : A + \frac{q}{2}B - \frac{q^2}{2.4}C + \frac{3q^3}{2.4.6}D - \frac{3.5q^4}{2.4.6.8}E$ &c. *Q.E.D.*

$= \frac{2500}{810000} = \frac{1}{324}$, $\sqrt{cc+yy} = 10\sqrt{10} = 31{\cdot}62277662$, and $\frac{y+\sqrt{cc+yy}}{c} = 1{\cdot}387425887$, whoſe hyp. log. is ${\cdot}3274501 = A$; alſo $\frac{y\sqrt{cc+yy} - ccA}{2} = 10{\cdot}76133 = B$, $\frac{y^3\sqrt{cc+yy} - 3ccB}{4} = 641{\cdot}796405 = C$, $\frac{y^5\sqrt{cc+yy} - 5ccC}{6} = 45698{\cdot}97\tfrac{1}{3} = D$, and $\frac{y^7\sqrt{cc+yy} - 7ccD}{8} = 3540529{\cdot}3125 = E$.

$$\begin{array}{lrl@{\qquad}rl}
\text{Then} + & A = & {\cdot}327450 & -\frac{q^2}{2.4}C = & {\cdot}000764 \\
+ & \frac{q}{2}B = & {\cdot}016607 & -\frac{3.5q^4}{2.4.6.8}E = & {\cdot}000012 \\
 & +\frac{3q^3}{2.4.6}D = & {\cdot}000084 & & \overline{-{\cdot}000776} \\
 & + & \overline{{\cdot}344141} & & \\
 & - & {\cdot}000776 & & \\
 & & \overline{{\cdot}343365} & &
\end{array}$$

$= $ the ſum of the ſeries,
which multip. by $c = 30$
produces $10{\cdot}30095 =$ the length of the arc required.

*RULE II.

Uſing the ſame ſymbols as in the laſt rule, the arc will be expreſſed by the ſeries $y \times : 1 + \frac{a^2}{6c^4}y^2 - \frac{a^4+4a^2c^2}{40c^8}y^4 + \frac{a^6+4a^4c^2+8a^2c^4}{112c^{12}}y^6 - \frac{5a^8+24a^6c^2+48a^4c^4+64a^2c^6}{1152c^{16}}y^8$ &c.

or

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* DEMONSTRATION.

For, by the demonſt. of the laſt rule, $\dot{z}$ is $= \dot{y}\sqrt{\frac{cc+\frac{aa+cc}{cc}yy}{cc+yy}}$ = (by extracting the root of the numerator and denominator, and then dividing the one by the other) $\dot{y} \times : 1 + \frac{a^2}{2c^4}y^2 - \frac{a^4+4a^2c^2}{2.4c^8}y^4 + \frac{a^6+4a^4c^2+8a^2c^4}{4.4c^{12}}y^6 - \frac{5a^8+24a^6c^2+48a^4c^4+64a^2c^6}{4.4.8c^{16}}y^8$ &c. And, by taking the fluents, we obtain the ſeries as in the rule.

or $y \times : 1 + \frac{a^2y^2}{6c^4}A - \frac{a^2+4c^2}{c^4} \cdot \frac{3y^2}{20}B + \frac{a^4+4a^2c^2+8c^4}{a^2+4c^2} \cdot \frac{5y^2}{14c^4}C - \frac{5a^6+24a^4c^2+48a^2c^4+64c^6}{a^4+4a^2c^2+8c^4} \cdot \frac{7y^2}{72c^4}D$ &c. putting A, B, C, &c. for the firſt, ſecond, third, &c. term.

EXAMPLE.

Taking the ſame example as to the laſt rule, in which $a = 40$, $c = 30$, and $y = 10$; we ſhall obtain

$$\begin{array}{ll}
A = 1 & C = \cdot 003170 \\
B = 0 \cdot 032922 & E = \underline{\quad 59} \\
D = \underline{\quad 398} & - \cdot 003229 \\
+ 1 \cdot 033320 & \\
- \underline{\cdot 003229} &
\end{array}$$

the ſum = 1·030091 which multiplied
by $y = 10$
produces 10·30091 = the arc, nearly the ſame as before.

RULE III.*

As the ſum of 15 times the parameter of the axe and a fourth proportional to the axe, the abſciſſa, and

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* DEMONSTRATION.

Firſt $x = \frac{a\sqrt{c^2+y^2}}{c} - a$ (uſing the ſame ſymbols as in the laſt rules) $= \frac{ay^2}{2c^2} - \frac{ay^4}{2.4c^4} + \frac{3ay^6}{2.4.6c^6} - \frac{3.5ay^8}{2.4.6.8c^8}$ &c.

Then proceed as in rule 5 for the elliptic arc.

Thus $y \times \frac{A+\overline{B+1}.x}{A+Bx} = y \times \frac{A+\overline{B+1} \times : \frac{ay^2}{2c^2} - \frac{ay^4}{8c^4} \text{ \&c.}}{A+B \times : \frac{ay^2}{2c^2} - \frac{ay^4}{8c^4} \text{ \&c.}} =$ $y \times : 1 + \frac{ay^2}{2Ac^2} - \frac{Aa+2Ba^2}{8A^2c^4}y^4$ &c. which put $= y \times : 1 + \frac{a^2y^2}{6c^4} - \frac{a^4+4a^2c^2}{40c^8}y^4$ &c. the value of the arc in the laſt rule. Then, by equating and the ſum of 9 times the axe and 21 times the parameter; is to the ſum of 15 times the parameter and a fourth proportional to the axe, the abſciſſa, and the ſum of 19 times the axe and 21 times the parameter; ſo is the ordinate, to the length of the curve, very nearly.

That is, $\dfrac{15p + \dfrac{19t+21p}{t}x}{15p + \dfrac{9t+21p}{t}x} \times y =$ the length of the curve nearly; where t is = the tranſverſe axe, $p =$ its parameter or a third proportional to the tranſverſe and conjugate, $x =$ the abſciſſa, and $y =$ the ordinate correſponding to the arc.

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equating the correſponding terms, we have $\dfrac{a}{2Ac^2} = \dfrac{a^2}{6c^4}$, or $A = \dfrac{3c^2}{a} = \frac{3}{2}p$; and $\dfrac{Aa + 2Ba^2}{8A^2c^4} = \dfrac{a^4 + 4a^2c^2}{40c^8}$, hence $B = \dfrac{9a^2 + 21c^2}{10aa} = \dfrac{9a + \dfrac{21cc}{a}}{10a} = \dfrac{9t + 21p}{10t}$, putting p for the parameter and t for the whole tranſverſe axe.

Conſequently $y \times \dfrac{A + \overline{B+1}.x}{A + Bx}$ is $= y \times \dfrac{\frac{3}{2}p + \dfrac{19t + 21p}{10t}x}{\frac{3}{2}p + \dfrac{9t + 21p}{10t}x} = y \times \dfrac{15p + \dfrac{19t + 21p}{t}x}{15p + \dfrac{9t + 21p}{t}x}$. *Q.E.D.*

COROLLARY.

A right line may be found nearly equal to an hyperbolic arc by proceeding in the ſame manner as in the corollary to rule 5 for the arc of the ellipſe, taking here $PG = \frac{3}{2}AP + \dfrac{19AK + 21AP}{10AK} \times AD$:

EXAMPLE.

Taking, again, the ſame example, we have $t = 80$, $p = \frac{60 \times 60}{60} = \frac{360}{8} = 45$, $y = 10$, and $x = \frac{a\sqrt{c^2 + y^2}}{c} - a = a \times \frac{\sqrt{c^2 + y^2} - c}{c} = 40 \times \frac{\sqrt{1000} - 30}{30} = 4 \times \frac{10\sqrt{10} - 30}{3} = 2{\cdot}16370216$.

Then $y \times \frac{15p + \frac{19t + 21p}{t} x}{15p + \frac{9t + 21p}{t} x} =$

$$10 \times \frac{15 \times 45 + \frac{19 \times 80 + 21 \times 45}{80} \times 2{\cdot}16370216}{15 \times 45 + \frac{9 \times 80 + 21 \times 45}{80} \times 2{\cdot}16370216} = 10 \times \frac{15 \times 9 \times 80 + \overline{19 \times 16 + 21 \times 9} \times 2{\cdot}16370216}{15 \times 9 \times 80 + \overline{9 \times 16 + 21 \times 9} \times 2{\cdot}16370216} = 10 \times \frac{11866{\cdot}70516488}{11520{\cdot}51281928} =$$

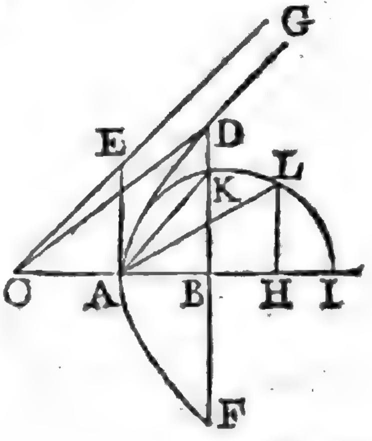
$10{\cdot}3005 =$ the length of the arc required, very near.

PROBLEM VI.

To find the area of a ſegment cut off an hyperbola by a double ordinate.

RULE I.*

From the product of the ordinate and its diſtance from the center, ſubtract the product ariſing from the multiplication of the product of the ſemi-axes, by the hyperbolic logarithm of the quotient reſulting from the diviſion of the ſum of the products



ducts of the ſemi-tranſverſe multiplied by the ordinate and the ſemi-conjugate multiplied by the ſaid diſtance of the ordinate from the center, by the product of the ſemi-axes; and the remainder will be the area cut off by the double ordinate.*

That

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* DEMONSTRATION.

The fluxion of the area ABD is $= y\dot{v} =$ (by the nature of the curve) $\frac{c\dot{v}\sqrt{vv-aa}}{a}$, the fluent of which is $= \frac{cv\sqrt{vv-aa}}{2a} - \frac{ac}{2} \times$ h. l. of $\frac{v+\sqrt{vv-aa}}{a} =$ the area ABD, the double of which will be $\frac{cv\sqrt{vv-aa}}{a} - ac \times$ h. l. of $\frac{v+\sqrt{vv-aa}}{a} = vy - ac \times$ h. l. of $\overline{\frac{v}{a}+\frac{y}{c}} = vy - ac \times$ h. l. of $\frac{ay+cv}{ac} =$ the whole ſegment DAF. Q.E.D.

COROLLARY I.

The ſame ſeg. DAF or $\frac{cv\sqrt{vv-aa}}{a} - ac \times$ h. l. of $\frac{v+\sqrt{vv-aa}}{a}$ is, alſo, $= \frac{ay\sqrt{cc+yy}}{c} - ac \times$ h. l. of $\frac{y+\sqrt{cc+yy}}{c} = ac \times$ $\overline{q\sqrt{1+qq} - \text{h. l. of } q+\sqrt{1+qq}} =$ $ac \times \overline{Q\sqrt{QQ-1} - \text{h. l. of } Q+\sqrt{QQ-1}}$, putting $q = \frac{y}{c}$, and $Q = \frac{v}{a}$.

COROLLARY II.

If from the triangle $CBD = \frac{1}{2}vy$, be taken the ſemi-ſegment $ABD = \frac{1}{2}vy - \frac{1}{2}ac \times$ h. l. of $\frac{ay+cv}{ac}$, there will remain $\frac{1}{2}ac \times$ h. l. of $\frac{ay+cv}{ac}$ for the value of the hyperbolic ſector CAD. When y or v are ſuppoſed infinite; then the ſector becomes the ſpace CAG

between

That is, $CB \times BD - CA \times AE \times$ hyp. log. of $\frac{CA \times BD + AE \times CB}{CA \times AE} = vy - ac \times$ hyp. log. of $\frac{ay + cv}{ac} =$ the area ADF, putting $a =$ the ſemi-tranſverſe axe CA, $c = AE$ the ſemi-conjugate, $y = BD$ the ordinate, and $v = BC$ its diſtance from the center.

Note

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between the aſymptote and the curve infinitely produced, and the expreſſion for it becomes infinite.

COROLLARY III.

When the hyperbola is equilateral, then a is $= c$, and the area $DAF = vy - aa \times$ h. l. of $\frac{v+y}{a}$.

COROLLARY IV.

When A and C coincide, the hyperbola becomes a triangle, and becauſe CA and AE are then each $= 0$, the rule will become barely $CB \times BD$ or $AB \times BD$ for the area.

COROLLARY V.

Similar ſegments of any hyperbolas are to each other as the rectangles of their axes. (Where by ſimilar ſegments are meant thoſe whoſe ordinates or abſciſſas are as their like axes or diameters.) For, by cor. 1, the ſeg. being expreſſed by $ac \times \overline{q\sqrt{1+qq} - \text{h. l. of } q + \sqrt{1+qq}}$ or $ac \times Q\sqrt{QQ-1} - \text{h. l. of } Q + \sqrt{QQ-1}$, and $q = \frac{y}{c}$, or $Q = \frac{v}{a}$, being the ſame in all ſimilar ſegments, thoſe ſegments will be as ac.——And hence, ſimilar ſegments of equilateral hyperbolas are as the ſquares of their axes, or of their like diameters.

Hence, alſo, hyperbolas of the ſame tranſverſe axe and abſciſſa, are to one another as their conjugate axes; but if their baſes or ordinates and conjugate axes be the ſame, they will be as their tranſverſe axes. For when a and v are the ſame, the quantity $\frac{cv\sqrt{vv-aa}}{a} - ac \times$ h. l. of $\frac{v+\sqrt{vv-aa}}{a}$ will be as c; and if c and y be the ſame, the quantity $\frac{ay\sqrt{cc+yy}}{c} - ac \times$ h. l. of $\frac{y+\sqrt{cc+yy}}{c}$ will be as a.

And

Note 1. The ſame rules will ſerve for any other two conjugate diameters and their abſciſſa and ordinate; multiplying the reſult by the ſine of the angle formed by the abſciſſa and ordinate, to the radius 1.

Note 2. The hyperbolic logarithm is equal to the common logarithm multiplied by 2·302585093.

Ex-

And, conſequently, having the quadrature of any one hyperbola, from it we may find that of any other hyperbola; viz. knowing the the area anſwering to every abſciſſa in any one hyperbola, we can find the area anſwering to any abſciſſa in any other hyperbola. For we can find a ſimilar abſciſſa in the ſquared hyperbola, with the correſponding area. Then, as the rectangle of the axes of the ſquared hyperbola is to the rectangle of thoſe of the hyperbola propoſed, ſo is the area of the former to that of the latter.——Thus, let t, c be the tranſverſe and conjugate axes of any propoſed hyperbola, and x its abſciſſa; alſo T, C the axes of the ſquared hyperbola: Then, $t : T :: x : \frac{Tx}{t} =$ the ſimilar abſciſſa in the ſquared hyperbola, whoſe correſponding area call A: Then, as $TC : A :: tc : \frac{tcA}{TC} =$ the area of the hyperbola propoſed.——Now, it will be moſt convenient to have the ſquared hyperbola an equilateral one whoſe axes are each an unit. For then we need only divide the abſciſſa of the hyperbola propoſed by its tranſverſe axe, and multiply the product of its axes by the area anſwering to this quotient in the ſquared or equilateral hyperbola. And this is ſimilar to the finding of the ſegments of circles and ellipſes from thoſe of a circle whoſe ſegments are already calculated and ranged in tables.

COROLLARY VI.

If a and c be any other two conjugate diameters, and s the ſine of the angle formed by a and its ordinates, to the radius 1; it is extremely evident that $\frac{scv\sqrt{vv-aa}}{a} - sac \times$ h. l. of $\frac{v+\sqrt{vv-aa}}{a}$ will denote the oblique ſegment cut off by a double ordinate to the diameter a. And if, in this expreſſion, for sa and sv be ſubſtituted reſpec-

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EXAMPLE.

Required the area of an hyperbola whoſe baſe is 24 and altitude 10, the tranſverſe axe being 30.

Here $a = 15$, $y = 12$, and $v = 25$; hence $c = \frac{ay}{\sqrt{vv - aa}} = \frac{15 \times 12}{\sqrt{40 \times 10}} = \frac{180}{20} = 9$.

5 A Then

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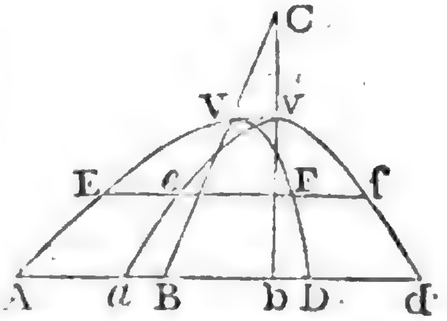
reſpectively A and V, the ſaid oblique hyperbola will be expreſſed by $\frac{cV\sqrt{VV - AA}}{A} - Ac \times$ h. l. of $\frac{V + \sqrt{VV - AA}}{A}$ which alſo expreſſes a right hyperbola whoſe tranſverſe and conjugate axes are A, c, and the diſtance of its baſe from the center V.

And hence is evident the following conſtruction which SCHOOTEN firſt demonſtrated, in a very operoſe manner, in his *Exercitationes Mathematicæ Lib.* 4.

Viz. Let AD be a double ordinate to the ſemi-diameter CV of the oblique hyperbola AVD. Draw Vv parallel and Cvb perpendicular to AD; and with the ſemi-tranſverſe Cv and ſemi-conjugate = the ſemi-conjugate of CV deſcribe the right hyperbola avd. And the right hyperbola avd will be = AVD the oblique one propoſed; alſo $AD = ad$, and every other correſponding ordinates EF, ef, equal to each other.



COROLLARY VII.

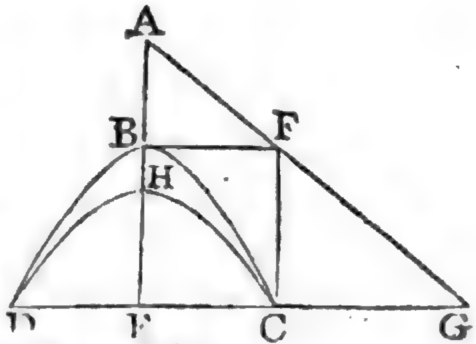
Hence it appears that all hyperbolas having the ſame center and equal baſes, and between the ſame parallels Ad, Vv, infinitely produced, are equal to each other; as are, alſo, their correſponding ſections EF, ef, parallel to the baſes; and likewiſe the intercepted fruſtums $AEFD$, $aefd$.

Co-

Then $\frac{ay+cv}{ac} = \frac{15\times12+9\times25}{15\times9} = \frac{4+5}{3} = 3$, whoſe hyperbolic logarithm is 1·0986122888$\frac{2}{3}$, which being multiplied by $ac = 15\times9$, produces 148·31265897; and this being taken from $vy = 25\times12 = 300$, leaves 151·68734103 = the area required.

RULE

COROLLARY VIII.



Let AB be the ſemi-tranſverſe axe of the hyperbola CBD; draw CF parallel and FB perpendicular to BA, and draw AFG meeting DC, produced, in G; then upon the baſe CD of the hyperbola deſcribe a parabola CHD having its parameter equal to the conjugate axe of the hyperbola. And $\overline{EG-CH} \times AB$ will be equal to the hyperbolic area BCE.

For the area $BCE = A = \frac{1}{2}vy - \frac{1}{2}ac \times$ h. l. of $\frac{ay+cv}{ac}$, and $a \times$ the curve $CH = aC = \frac{1}{2}vy + \frac{1}{2}ac \times$ h. l. of $\frac{ay+cv}{ac}$, by addition, $A + aC = vy$, and therefore $A = vy - aC = a \times \overline{\frac{vy}{a} - C} = \overline{\frac{AE \times EC}{AB} - CH} \times AB = \overline{EG - CH} \times AB$.

So that the quadrature of the hyperbola depends upon the rectification of the parabola.

COROLLARY IX.

By ſubtracting the ſegment $vy - ac \times$ h. l. of $\frac{ay+cv}{ac}$ from the ſegment $VY - ac \times$ h. l. of $\frac{aY+cV}{ac}$, there will reſult $VY - vy - ac \times$ h. l. of $\frac{aY+cV}{ay+cv}$ for the fruſtum included by two parallel double ordinates $2Y$, $2y$, their diſtances from the center being V, v.

*RULE II.

Putting $x =$ the abſciſſa AB, and the reſt of the letters as in the firſt rule, alſo $z = \frac{x}{2a+x}$. Then $2xy \times : \frac{1}{3} - \frac{1}{1.3.5}z - \frac{1}{3.5.7}z^{2} - \frac{1}{5.7.9}z^{3} - \frac{1}{7.9.11}z^{4}$ &c. or $2xy \times : \frac{1}{3} - \frac{1}{5}Az - \frac{1}{7}Bz - \frac{3}{9}Cz - \frac{5}{11}Dz$ &c. will be the area of the ſemi-ſegment ABD. (Fig. to rule 1) putting A, B, C, &c. for the firſt, ſecond, third, &c. term.

Ex-

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* DEMONSTRATION.

The fluxion of the area is $= y\dot{x} = \frac{c\dot{x}\sqrt{2ax+xx}}{a}$. Then inſtead of $\frac{x}{2a+x}$ write z, in order that the fluent quantity z may be leſs than 1, and by that means the ſeries be made to converge by its powers, and the ſaid fluxion $y\dot{x} = \frac{c\dot{x}\sqrt{2ax+xx}}{a}$ will be $= \frac{4acz^{\frac{1}{2}}\dot{z}}{\overline{1-z}|^{3}}$ $= 2acz^{\frac{1}{2}}\dot{z} \times : 1.2 + 2.3z + 3.4z^{2} + 4.5z^{3}$ &c. and the fluent is $4acz^{\frac{1}{2}} \times : \frac{1.2}{3} + \frac{2.3}{5}z + \frac{3.4}{7}z^{2} + \frac{4.5}{9}z^{3}$ &c. $=$ the ſemi-ſegment ABD.

Now although this ſeries converge by the powers of z, yet the coefficients actually diverge; in order, then, to make the coefficients to converge, let the ſeries be multiplied by ſome convenient quantity leſs than 1, as $1-z$, and the quantity $4acz^{\frac{1}{2}}$ divided by the ſame, and the reſult will be $\frac{8acz^{\frac{1}{2}}}{1-z} \times : \frac{1.1}{1.3} + \frac{2.2}{3.5}z + \frac{3.3}{5.7}z^{2} + \frac{4.4}{7.9}z^{3}$ &c. for the ſaid area ABD. And, to make the ſeries converge ſtill quicker, let the ſame operation be performed upon this ſeries, and there will reſult $\frac{8acz^{\frac{1}{2}}}{\overline{1-z}|^{2}} \times : \frac{1}{3} - \frac{1}{1.3.5}z - \frac{1}{3.5.7}z^{2} - \frac{1}{5.7.9}z^{3} - \frac{1}{7.9.11}z^{4}$ &c. for the value of the ſaid area.

But

EXAMPLE.

Taking here the laſt example, in which $x = 10$, $y = 12$, and $2a = 30$; z will be $= \frac{x}{2a+x} = \frac{10}{30+10} = \frac{10}{40} = \frac{1}{4}$.

Then

But $\frac{8acz^{\frac{3}{2}}}{\overline{1-z}|^{2}} = \frac{2cx\sqrt{2ax+xx}}{a} = 2xy$; wherefore $2xy \times : \frac{1}{3} - \frac{1}{1.3.5}z - \frac{1}{3.5.7}z^{2} - \frac{1}{5.7.9}z^{3}$ &c. $=$ the area *ABD*. *Q.E.D.*

COROLLARY I.

An hyperbola is always leſs than a parabola of the ſame baſe and altitude; for $2xy \times : \frac{1}{3} - \frac{1}{1.3.5}z - \frac{1}{3.5.7}z^{2}$ &c. is always leſs than $2xy \times \frac{1}{3} = \frac{2}{3}xy =$ the parabola; but as the altitude is diminiſhed, the hyperbola approaches nearer and nearer to the meaſure of the parabola, till, at laſt, they vaniſh in a ratio of equality.

COROLLARY II.

An hyperbola, is, evidently, for any finite altitude, always greater than a triangle of the ſame baſe and altitude; but as the altitude is increaſed, the triangle approaches nearer and nearer to the value of the hyperbola, till, at length, when the altitude is infinite, they become accurately equal to each other. Which I demonſtrate, by the method of increments, after the notation of the ingenious *Mr Emerſon*, thus:

When the altitude x is infinite, then $z = \frac{x}{2a+x} = \frac{x}{x} = 1$, and the general ſeries expreſſing the value of the area will become $2xy \times : \frac{1}{3} - \frac{1}{1.3.5} - \frac{1}{3.5.7} - \frac{1}{5.7.9}$ &c.——Now to find the ſum s of any number n of the negative terms; ſince it is evident that the n term is $\frac{1}{\overline{2n-1}.\overline{2n+1}.\overline{2n+3}}$, or, putting $v = 2n-1$, then $\underset{\cdot}{v} = 2\underset{\cdot}{n} = 2$,

and

Then $A = \frac{1}{3} = {}^{\cdot}33$ &c.
$B = \frac{1}{5}Az = {}^{\cdot}01666667$
$C = \frac{1}{7}Bz = 51524$
$D = \frac{3}{9}Cz = 4960$
$E = \frac{5}{11}Dz = 5639$
$F = \frac{7}{13}Ez = 76$
$G = \frac{9}{15}Fz = 11$
$H = \frac{11}{17}Gz = 2$

the ſum of theſe negative terms is ·0173̇18039 which being taken from the firſt term $\frac{1}{3}$, leaves ·316015294; and this multiplied by $2xy = 240$, produces $75{}^{\cdot}84367056$ = the ſemi-ſegment, the double of which is $151{}^{\cdot}68734112$ = the area required.

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the n term $= \frac{1}{v.\overline{v+2}.\overline{v+3}} = \frac{1}{vv_1v_2}$; the $\overline{n+1}$ term or $\dot{s}$ will be $\frac{1}{v_1v_2v_3}$; and the integral $s = A - \frac{1}{2vv_1v_2} = A - \frac{1}{4v_1v_2}$: But when n is $= 1$, then s ought to be $\frac{1}{vv_1v_2} =$ (in this caſe) $A - \frac{1}{4v_1v_2}$, hence $A = \frac{1}{4v_1v_2} + \frac{1}{vv_1v_2} = \frac{v+4}{4vv_1v_2} = \frac{1+4}{4.1.3.5} = \frac{5}{60} = \frac{1}{12}$; and conſequently the correct integral is $s = \frac{1}{12} - \frac{1}{4v_1v_2} = \frac{1}{12} - \frac{1}{4.\overline{2n+1}.\overline{2n+3}}$, which when n is infinite becomes accurately $= \frac{1}{12}$, that is, the ſum of the infinite ſeries $\frac{1}{1.3.5} - \frac{1}{3.5.7}$ &c. is $= \frac{1}{12}$; which being taken from $\frac{1}{3}$ leaves $\frac{1}{3} - \frac{1}{12} = \frac{1}{4}$, and hence $2xy \times \frac{1}{4} = \frac{1}{2}xy =$ the area of the hyperbola when x is infinite, and is therefore equal to a triangle of the ſame baſe and altitude.

COROLLARY III.

Hence the ſpace included between the arc of an hyperbola, infinitely produced, and its chord, is of a finite magnitude, although it be

RULE III.

Take $BH = \frac{3AB^2}{8AC}$ = to three-fourths of a third proportional to the tranſverſe axe $2AC$, and the abſciſſa AB; upon the diameter AI = the parameter of the axe, deſcribe a circle meeting with BD in K, and with HL, parallel to BD, in L; and draw AK, AL. [See fig. to rule 1.]

Then will the ſegment FAD be nearly equal to $\frac{4AL + AD}{15} \times 4AB$, or $= \frac{4\sqrt{tx + \frac{1}{4}xx} + \sqrt{tx}}{15} \times \frac{4cx}{t}$, putting

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be of an infinite length. For this ſpace is the difference between the hyperbola and triangle, which, as above, is nothing, that is, nothing in compariſon with an infinite ſpace.

COROLLARY IV.

By cor. 3 rule 2 for the ſegment of a circle, the area of a circle whoſe diameter is 1, is $n = 2 \times : \frac{1}{3} + \frac{1}{1.3.5} - \frac{1}{3.5.7} +$ &c. and, by cor. 2 to this problem, $\frac{1}{2} = 2 \times : \frac{1}{3} - \frac{1}{1.3.5} - \frac{1}{3.5.7} -$ &c. from hence are deduced the following equations:

I. $n = \frac{5}{6} - 4 \times : \frac{1}{3.5.7} + \frac{1}{7.9.11} + \frac{1}{11.13.15} + \frac{1}{15.17.19}$ &c.

II. $n = \frac{1}{2} + 4 \times : \frac{1}{1.3.5} + \frac{1}{5.7.9} + \frac{1}{9.11.13} + \frac{1}{13.15.17}$ &c.

III. $n = \frac{2}{3} + 12 \times : \frac{1}{1.3.5.7} + \frac{1}{5.7.9.11} + \frac{1}{9.11.13.15}$ &c.

IV. $\frac{1}{96} = \frac{1}{1.3.5.7} + \frac{2}{5.7.9.11} + \frac{3}{9.11.13.15}$ &c.

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ting t = the tranſverſe $2AC$, c = the conjugate, and $x = AB$ the abſciſſa.*

EXAMPLE.

Taking the ſame example, in which $t = 30$, $c = 18$, and $x = 10$; we ſhall have $\frac{\sqrt{tx} + 4\sqrt{tx + \frac{3}{4}x^2}}{15} \times \frac{4cx}{t} = \frac{\sqrt{300} + 4\sqrt{375}}{450} \times 720 = \frac{10\sqrt{3} + 20\sqrt{15}}{5} \times 8 = \overline{\sqrt{3} + 2\sqrt{15}} \times 16 = 151{\cdot}64828$ = the area nearly.

RULE

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* DEMONSTRATION.

By proceeding as in rule 2 for the circular ſegment, we obtain $\frac{cx\sqrt{tx}}{t} \times : \frac{2}{3} + \frac{x}{5t} - \frac{x^2}{28t^2}$ for the true value of the ſemi-ſegment ADB. Then, after the method uſed in demonſtrating rule 6 for the cir. ſegment, ſuppoſe ADB to be $= \frac{cmx\sqrt{tx + \frac{3}{4}xx}}{t} + \frac{cnx\sqrt{tx}}{t} = \frac{cx\sqrt{tx}}{t} \times : m + n + \frac{3mx}{8t}$ &c. hence, comparing the like terms, $m + n = \frac{2}{3}$, and $\frac{3m}{8} = \frac{1}{5}$; and therefore $m = \frac{8}{15}$, and $n = \frac{2}{3} - \frac{8}{15} = \frac{2}{15}$. Which values being ſubſtituted for them, we have $ADB =$

$$\frac{2cx}{t} \times \frac{4\sqrt{tx + \frac{3}{4}xx} + \sqrt{tx}}{15} = 2x \times \frac{4\sqrt{\frac{c^2x}{t} + \frac{3c^2x^2}{4tt}} + \sqrt{\frac{c^2x}{t}}}{15} =$$

$$= 2AB \times \frac{4\sqrt{\overline{AB + \frac{3AB^2}{8AC}} \times AI} + \sqrt{BA \times AI}}{15} = 2AB \times$$

$$\frac{4\sqrt{HA \times AI} + \sqrt{BA \times AI}}{15} = 2AB \times \frac{4AL + AD}{15}. \quad Q.E.D.$$

And this rule was firſt given by *Sir I. Newton*, but without demonſtration.

RULE IV.

If BH be taken $= \frac{5xx}{7t} = \frac{5AB^2}{14AC} =$ to five-ſevenths of the ſaid third proportional to the axe $2AC$, and the abſciſſa AB; and the lines drawn as in the laſt rule.

Then will $\frac{21AL+4AK}{75} \times 4AB$, or $\frac{21\sqrt{tx+\frac{5}{7}xx}+4\sqrt{tx}}{75} \times \frac{4cx}{t}$ expreſs the area nearer than by the laſt rule.*

Ex-

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* DEMONSTRATION.

Taking $BH = \frac{Axx}{t}$, let, as before, the area be expreſſed by $AB \times \overline{m \times AL + n \times AK} = AB \times \overline{m\sqrt{HA \times AI} + n\sqrt{BA \times AI}} = mx\sqrt{x+\frac{Axx}{t}} \times \frac{cc}{t} + nx\sqrt{x \times \frac{cc}{t}} = \overline{m\sqrt{tx+Axx}+n\sqrt{tx}} \times \frac{cx}{t} = \frac{cx\sqrt{tx}}{t} \times : m+n+\frac{mAx}{2t} - \frac{mA^2x^2}{8tt}$ &c. which being compared with the true ſeries $\frac{cx\sqrt{tx}}{t} \times : \frac{2}{3} + \frac{x}{5t} - \frac{x^2}{28t^2}$ &c. we obtain theſe three equations, $m+n=\frac{2}{3}$, $\frac{mA}{2}=\frac{1}{5}$, and $\frac{mA^2}{8}=\frac{1}{28}$; whence $A=\frac{5}{7}$, $m=\frac{14}{25}$, and $n=\frac{8}{75}$. Conſequently $BH=\frac{Axx}{t} = \frac{5xx}{7t}$, and the ſemi-ſegment $= 2AB \times \frac{21AL+4AK}{75} = \frac{2cx}{t} \times \frac{21\sqrt{tx+\frac{5}{7}xx}+4\sqrt{tx}}{75}$. Q.E.D.

Which likewiſe is another rule given by *Sir I. Newton*, without demonſtration.

EXAMPLE.

Taking, ſtill, the ſame example, we have $\frac{21\sqrt{tx+\frac{5}{7}xx}+4\sqrt{tx}}{75}\times\frac{4cx}{t}=\frac{21\sqrt{371\frac{3}{7}}+4\sqrt{300}}{75}\times\frac{720}{30}=\frac{30\sqrt{182}+40\sqrt{3}}{75}\times 24=\frac{3\sqrt{182}+4\sqrt{3}}{5}\times 16=151{\cdot}68133$ = the area nearly.

PROBLEM VII.

To find the area of the fruſtum of an hyperbola, included between two parallel double ordinates.

RULE.

Multiply the ſemi-tranſverſe by the one ordinate, and its diſtance from the center by the ſemi-conjugate, and take the ſum of the products; do the ſame by the other ordinate and its diſtance from the center, taking the ſum of the products alſo; divide the greater ſum by the leſs, and multiply the hyperbolic logarithm of the quotient by the product of the ſemi-axes; then ſubtract this laſt product from the difference of the products ariſing from the multiplication of each ordinate by its diſtance from the center, and the remainder will be the area required.

That is, $VY-vy-ac\times$ hyp. log. of $\frac{aY+cV}{ay+cv}=$ the area included by $2Y$ and $2y$, V and v being their reſpective diſtances from the center, and a and c the ſemi-axes. As is proved in cor. 8 to rule 1 to the laſt problem.

Or, if there be calculated the two ſegments whoſe baſes are $2Y$ and $2y$, their difference will be the area ſought.

EXAMPLE.

To find the area of the fruſtum whoſe altitude is 5; its leſs end being, as in the example to the laſt problem, 24, its diſtance from the center of the hyperbola 25, the tranſverſe axe 30, and the conjugate 18.

Here $a = 15$, $c = 9$, $v = 25$, $y = 12$, and $V = 30$; hence $Y = \frac{c\sqrt{VV - aa}}{a} = \frac{9\sqrt{30^2 - 15^2}}{15} = 9\sqrt{2^2 - 1^2} = 9\sqrt{3} = 15{\cdot}58845727$.

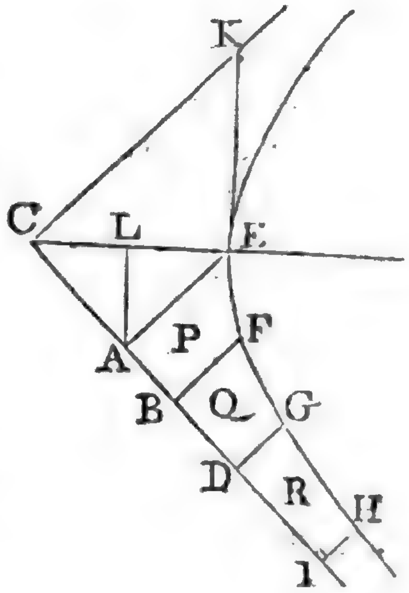
Then $\frac{aY + cV}{ay + cv} = \frac{15 \times 9\sqrt{3} + 9 \times 30}{15 \times 12 + 9 \times 25} = \frac{9\sqrt{3} + 3 \times 6}{12 + 3 \times 5} = \frac{\sqrt{3} + 2}{3} = 1{\cdot}244016936$, whoſe hyperbolic logarithm is ${\cdot}2183456$, which multiplied by $ac = 15 \times 9$ will produce $29{\cdot}476656$.

And $VY - vy = 30 \times 9\sqrt{3} - 25 \times 12 = 30 \times \overline{9\sqrt{3} - 10} = 30 \times 5{\cdot}58845727 = 167{\cdot}653718$.

Wherefore $167{\cdot}653718 - 29{\cdot}476656 = 138{\cdot}177062$ = the area required.

PROBLEM VIII.

To find the area included bebetween two ordinates BF, DG, *to an aſymptote* CD *of an hyperbola, drawn parallel to the other aſymptote.*



RULE.

From the vertex *E* of the curve draw *EA* parallel to *BF*.

Then the ſquare of *CA* or of *AE* multiplied by the hyperbolic logarithm of the quotient ariſing from the diviſion of *CD* by *CB*, or from the diviſion of *BF* by *DG*, will produce the area of *BFGD*.*

Ex-

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* DEMONSTRATION.

Put $CA = a$, $AB = x$, and $BF = y$; then, by the property of the curve, $aa = CB \times BF = \overline{a+x} \times y$, and hence $y = \frac{aa}{a+x}$, and $y\dot{x} = \frac{aa\dot{x}}{a+x}$; whoſe fluent is $aa \times$ h. l. of $\frac{a+x}{a} = CA^2 \times$ h. l. of $\frac{CB}{CA}$ = the area $ABFE$.

In like manner $ADGE$ is $= CA^2 \times$ h. l. of $\frac{CD}{CA}$.

And, taking the difference, we have $BDGF = CA^2 \times \overline{\text{h. l. of } \frac{CD}{CA} - \text{h. l. of } \frac{CB}{CA}} = CA^2 \times \text{h. l. of } \frac{CD}{CB}$; or by writing $\frac{CA^2}{BF}$ for CB, and $\frac{CA^2}{DG}$ for CD, the ſame area will be $= CA^2 \times$ h. l. of $\frac{BF}{DG}$. *Q. E. D.*

COROLLARY I.

If CB, CD, CI, &c. be in geometrical progreſſion; then $\frac{CD}{CB}$ will be $= \frac{CI}{CD}$, &c. and conſequently the ſpaces P, Q, R, &c. equal to one another; or the ſpaces $ABFE$, $ADGE$, $AIKE$, &c. a ſeries of arithmeticals, and are the logarithms of the geometricals $\frac{CB}{CA}$, $\frac{CD}{CA}$, $\frac{CI}{CA}$, &c. to the modulus CA; or the ſaid ſpaces will form a ſcale of hyperbolic logarithms whoſe abſolute numbers are CB, CD, CI, &c. when CA is 1.

Co-

EXAMPLE.

Required the area $BDGF$ included by two ordinates BF, DG, whoſe lengths are 8 and 6; the ſemi-tranſverſe CE being 15, and the ſemi-conjugate $EK = 9$.

Draw AL perpendicular to CE; then becauſe EA is $= AC$, CL will be $= LE$; and in the ſimilar triangles ALC, CEK, becauſe CL is $= \frac{1}{2}CE$, CA will be $= \frac{1}{2}CK$, and ſo $CA^2 = \frac{1}{4}CK^2 = \frac{CE^2 + EK^2}{4} = \frac{15^2 + 9^2}{4} = 9 \times \frac{5^2 + 3^2}{4} = 9 \times \frac{17}{2} = \frac{153}{2}$.

But

COROLLARY II.

The ſpace $ABFE$ is alſo equal to $a^2 \times : \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4}$ &c.

For the fluxion of the area $y\dot{x} = \frac{aa\dot{x}}{a+x}$ is $= aa\dot{x} \times : \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4}$ &c. whoſe fluent is $a^2 \times : \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4}$ &c. $= CA^2 \times : \frac{AB}{CA} - \frac{AB^2}{2CA^2} + \frac{AB^3}{3CA^3}$ &c.

After the ſame manner $ADGE$ is $= CA^2 \times : \frac{AD}{CA} - \frac{AD^2}{2CA^2} + \frac{AD^3}{3CA^3}$ &c.

And, conſequently, by taking the difference, we obtain $BDGF = CA^2 \times : \frac{AD - AB}{CA} - \frac{AD^2 - AB^2}{2CA^2} + \frac{AD^3 - AB^3}{3CA^3}$ &c.

COROLLARY III.

Hence the hyperbolic logarithm of $\frac{a+x}{a}$ is $\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4}$ &c.

But the hyperbolic logarithm of $\frac{BF}{DG} = \frac{8}{6} = \frac{4}{3}$ is ·287682.

Wherefore ·287682 × $\frac{153}{2}$ = 22·007673 = the area required.

PROBLEM IX.

To find the curve ſurface of an hyperboloid.

RULE.

Let a and c be the ſemi-axes of the generating hyperbola, and v the diſtance of its baſe from the center. Alſo let $A = \frac{aa}{\sqrt{aa + cc}}$ be the ſemi-tranſverſe of another hyperbola whoſe ſemi-conjugate is c the ſame with that of the former. Then find, by problem 7, the area of the fruſtum of this latter hyperbola whoſe two ends are diſtant from the center by v and a; multiply this area by 3·14159 &c. the circumference of a circle whoſe diameter is 1, and the product will be the ſurface required.

That is, $p \times \overline{vY - ay - Ac \times \text{hyp. log. of } \frac{AY + cv}{Ay + ac}} =$ the ſurface required; p being = 3·14159 &c. and Y, y, the ordinates, of the latter hyperbola, whoſe diſtances from the center are v, a.*

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* DEMONSTRATION.

Put, here, w = the ordinate to the abſciſſa v, and z = the curve; then $a : c :: \sqrt{vv - aa} : \frac{c\sqrt{vv - aa}}{a} = w$, and $\dot{w} = \frac{cv\dot{v}}{a\sqrt{vv - aa}}$;

hence

EXAMPLE.

Required the curve ſurface of an hyperboloid whoſe altitude is 10, the tranſverſe and conjugate axes of the generating hyperbola being 30 and 18.

Here $a = 15$, $c = 9$, $A = \frac{aa}{\sqrt{aa+cc}} = \frac{15^2}{\sqrt{15^2+9^2}} = \frac{75}{\sqrt{34}}$,

$v = 25$, $Y = \frac{c\sqrt{vv - AA}}{A} = \frac{9\sqrt{25^2 - \frac{75^2}{34}}}{\frac{75}{\sqrt{34}}} = 15$, and y

$= \frac{c\sqrt{aa - AA}}{A} = \frac{9\sqrt{15^2 - \frac{75^2}{34}}}{\frac{75}{\sqrt{34}}} = \frac{27}{5} = 5\frac{2}{5}$.

Hence

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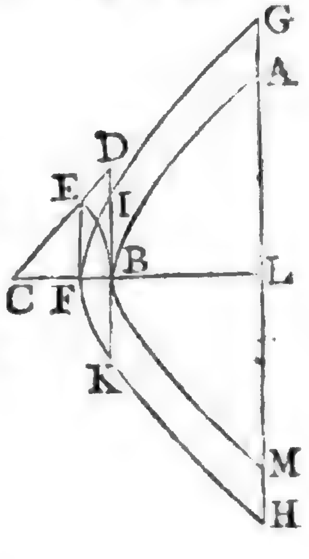
hence the fluxion of the ſurface will be $2pw\dot{z} = 2pw\sqrt{\dot{v}\dot{v} + \dot{w}\dot{w}}$ $= \frac{2pc\sqrt{vv-aa}}{a}\sqrt{\dot{v}\dot{v} + \frac{ccvv\dot{v}\dot{v}}{aa\times\overline{vv-aa}}} = \frac{2pc\dot{v}\sqrt{aa\times\overline{vv-aa}+ccvv}}{aa}$ $= \frac{2pc\dot{v}\sqrt{aa+cc}}{aa}\sqrt{vv - \frac{aaaa}{aa+cc}} = \frac{2pc\dot{v}\sqrt{vv-AA}}{A}$; which is, evidently, equal to p drawn into the fluxion of an hyperbolic ſegment, the ſemi-axes being A, c, and abſciſſa v. And by taking the correct fluent we obtain the expreſſion in the rule above given, which is the difference of two ſegments whoſe baſes are diſtant from the center by v and a; or the fruſtum whoſe two ends are diſtant from the center by the ſame quantities v and a. *Q. E. D.*

COROLLARY I.

Hence if ABM be the generating hyperbola, BC its ſemi-tranſverſe, BD perpendicular to BC and equal to the ſemi-conjugate. ——Draw CD; make $CE = CB$, and on CB let fall the perpendicular EF; with the ſemi tranſverſe CF and ſemi-conjugate of ABM deſcribe the hyperbola GFH.



Then

Hence $vY - ay = 25 \times 15 - 15 \times 5\frac{2}{5} = 19\frac{3}{5} \times 15 = 294$; $Ac = \frac{75 \times 9}{\sqrt{34}} = \frac{75 \times 9\sqrt{34}}{34} = 115{\cdot}7615451$;

and $\frac{AY + cv}{Ay + ca} = \frac{\frac{75 \times 15}{\sqrt{34}} + 9 \times 25}{\frac{75 \times 27}{5\sqrt{34}} + 9 \times 15} = \frac{5}{3} \times \frac{5 + \sqrt{34}}{3 + \sqrt{34}} = \frac{19 + 2\sqrt{34}}{15}$

$= 2{\cdot}0441269$; whoſe hyperbolic logarithm is $\cdot 7149706$; which multiplied by $Ac = 115{\cdot}7615451$ produceth $82{\cdot}7661049$, which being taken from $vY - ay = 294$ leaves $211{\cdot}2338951$, which multiplied by $3{\cdot}14159$ &c. produces $663{\cdot}610853$ = the curve ſurface required.

PRO-

Then, as the diameter of a circle is to its circumference, ſo is the fruſtum $GIKH$, included by the parallels BI, AM, produced, to the ſurface generated by AB.

For, by ſimilar triangles, $DC : CB :: EC : CF = A = \frac{BC^2}{CD} = \frac{BC^2}{\sqrt{CB^2 + BD^2}} = \frac{aa}{\sqrt{aa + cc}}$; and all the reſt is evident.

COROLLARY II.

Putting $Z = \frac{FL}{2CF + FL}$, and $z = \frac{FB}{2CF + FB}$; and proceeding as in rule 2 for the hyperbolic ſegment, we ſhall obtain the ſurface generated by BA equal to $3{\cdot}14159$ &c. drawn into the difference of the ſerieſes $2FL \times GH \times : \frac{1}{3} - \frac{1}{1.3.5}Z - \frac{1}{3.5.7}Z^2 - \frac{1}{5.7.9}Z^3$ &c. and $2FB \times IK \times : \frac{1}{3} - \frac{1}{1.3.5}z - \frac{1}{3.5.7}z^2 - \frac{1}{5.7.9}z^3$ &c.

PROBLEM X.

To find the ſolidity of an hyperboloid.

RULE I.*

From the ſum of the height of the ſolid and the tranſverſe axe of the generating hyperbola ſubtract $\frac{1}{3}$ of the ſaid height, and divide the remainder by 2 times the ſaid ſum; then multiply continually together

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* DEMONSTRATION.

Let t be $=$ the tranſverſe, and $c =$ the conjugate axe of the generating hyperbola, $x =$ its abſciſſa or the altitude of the ſolid, $y =$ the ordinate or radius of the baſe, and $p = 3{\cdot}14159$ &c.

Then $yy = cc \times \frac{tx + xx}{tt}$, and the fluxion of the ſolid or $pyy\dot{x}$ is $= pccx\dot{x} \times \frac{t + x}{tt}$, whoſe fluent is $pccxx \times \frac{\frac{1}{2}t + \frac{1}{3}x}{tt} = pxyy \times \frac{t + \frac{2}{3}x}{2t + 2x} = pxyy \times \frac{t + x - \frac{1}{3}x}{2t + 2x} =$ the meaſure of the ſolid $=$ $\frac{1}{2}$ baſe $\times$ altitude $\times \frac{t + x - \frac{1}{3}x}{t + x}$. *Q.E.D.*

COROLLARY I.

An hyperboloid is to a paraboloid of the ſame baſe and altitude, as $t + \frac{2}{3}x$ is to $t + x$; and therefore the former is always leſs than the latter by the quantity $\frac{1}{6}pxyy \times \frac{x}{t + x}$: which difference, when x is infinitely little or nothing, is nothing; and the hyperboloid approaches nearer and nearer to the paraboloid as the common altitude is diminiſhed, till at laſt they vaniſh in a ratio of equality; alſo when the altitude is very little the hyperboloid is equal to the paraboloid very nearly: but when x is infinite, the ſame difference $\frac{1}{6}pxyy \times \frac{x}{t + x}$ becomes barely $\frac{1}{6}pxyy$, and the value of the infinitely long hyperboloid is $\frac{1}{2}pxyy - \frac{1}{6}pxyy = \frac{1}{3}pxyy =$ a cone of the ſame baſe and altitude, to which meaſure the hyperboloid continually approaches, and when the altitude is very great it is equal to the cone very nearly.

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together, the quotient, the altitude of the ſolid, and the area of the baſe, and the product will be the ſolidity required.

That is, baſe × altitude × $\frac{t + a - \frac{1}{3}a}{2t + 2a}$ = $parr$ × $\frac{t + a - \frac{1}{3}a}{2t + 2a}$ = the ſolidity, putting a for the altitude, r the radius of the baſe, t the tranſverſe axe, and $p = 3{\cdot}14159$ &c.

COROLLARY II.

When x is $= nt$, the general expreſſion becomes $\frac{1}{2}$ baſe × altitude × $\frac{1 + \frac{2}{3}n}{1 + n}$, where n may be any number integral or fractional. When n is infinitely little, this expreſſion becomes $\frac{1}{2}$ baſe × altitude or = the paraboloid of the ſame baſe and altitude, as before; if n be infinitely great, it will become $\frac{1}{2}$ baſe × altitude × $\frac{2}{3}$, that is, $\frac{2}{3}$ of the paraboloid, or = the cone of the ſame baſe and altitude, as above; when n is $= 1$, or $t = x$, the expreſſion becomes $\frac{1}{2}$ baſe × altitude × $\frac{5}{6}$, or $\frac{5}{6}$ of the paraboloid. Moreover, when n is between 1 and 0, the hyperboloid is between $\frac{5}{6}$ and $\frac{6}{6}$ of the paraboloid; and when n is between 1 and infinite, the hyperboloid is between $\frac{5}{6}$ and $\frac{4}{6}$ of the paraboloid of the ſame baſe and altitude.

COROLLARY III.

If the generating hyperbola be equilateral, ſince t is then $= c$, yy will be $= tx + xx$, and hence $t = \frac{yy - xx}{x}$; which being ſubſtituted inſtead of it, the general expreſſion for the hyperboloid will become $px \times \frac{yy - \frac{1}{3}xx}{2}$, which is the ſame expreſſion as that for the ſpheric ſegment, differing only in the ſign of the latter term, it being $-\frac{1}{3}xx$ here and $+\frac{1}{3}xx$ there.

COROLLARY IV.

When t is $= 0$, the rule becomes $\frac{1}{3}$ baſe × altitude, as it ought; for when t is $= 0$, the hyperboloid becomes a cone.

EXAMPLE.

Required the content of an hyperboloid whoſe altitude is 10, the radius of its baſe 12, and the ordinate to the middle of the abſciſſa or height $3\sqrt{7}$.

Here, by prob. 3, we ſhall have $\frac{10^2 \times 3^2 \times 7 \backsim 5^2 \times 12^2}{5 \times 12^2 \backsim 10 \times 3^2 \times 7}$ $= \frac{70 - 40}{8 - 7} = \frac{30}{1} = 30 =$ the tranſverſe.

Then $parr \times \frac{t + a - \frac{1}{3}a}{2t + 2a} = p \times 10 \times 144 \times \frac{40 - \frac{10}{3}}{80} =$ $p \times 1440 \times \frac{3\frac{2}{3}}{8} = p \times 1440 \times \frac{11}{24} = 3{\cdot}14159$ &c. $\times 660$ $= 2073{\cdot}451151369 =$ the content required.

RULE II.

To the ſquare of the radius of the baſe add the ſquare of the diameter in the middle between the baſe and top, multiply the ſum by the altitude, the product by 3·14159 &c. and $\frac{1}{6}$ of the laſt product will be the content of the ſegment.

That is, $\frac{rr + dd}{6} \times ap =$ the content of the ſegment, putting r and d for the radius of the baſe and diameter in the middle, a for the altitude, and $p = 3{\cdot}14159$ &c.—This is proved in corollary 1 to rule 2 for the next problem.

EXAMPLE.

Taking here the laſt example, in which $r = 12$, $d = 6\sqrt{7}$, and $a = 10$;

We ſhall have $\frac{12^2 + 36 \times 7}{6} \times 10p = 11 \times 6 \times 10p =$ $660p =$ the content the ſame as before.

PRO-

PROBLEM XI.

To find the content of the fruſtum of an hyperboloid.

* RULE I.

From the ſum of the ſquares of the ſemi-diameters of the two ends ſubtract $\frac{1}{3}$ of a fourth proportional to the ſquare of the tranſverſe, the ſquare of the conjugate, and the ſquare of the altitude of the fruſtum; multiply the remainder by the ſaid altitude, and the product by 3·14159 &c. and the half of the laſt product will be the content of the fruſtum.

That is, $\overline{DD + dd - \frac{aacc}{3tt}} \times \frac{1}{2}pa$ = the content, putting D and d for the ſemi-diameters of the ends, a = the altitude, t = the tranſverſe, c = the conjugate, and p = 3·14159 &c.

Ex-

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* DEMONSTRATION.

Uſing here y, Y for the ordinates or ſemi-diameters of the ends, x for the altitude; and putting A for the diſtance of the leſs ordinate y from the vertex of the whole ſolid; ſince YY is $= \frac{\overline{t+A+x} \times \overline{A+x}}{tt} \times cc$, we ſhall have the fluxion of the ſolid $\dot{s} = pYY\dot{x} = pcc\dot{x} \times \frac{At + AA + 2Ax + tx + xx}{tt}$, and the fluents give $s = pccx \times \frac{At + AA + Ax + \frac{1}{2}tx + \frac{1}{3}xx}{tt}$; and this, by ſubſtituting $\frac{yy}{cc}$ for $\frac{At + AA}{tt}$, and $\frac{YY}{cc}$ for $\frac{At + AA + 2Ax + tx + xx}{tt}$ becomes $\overline{YY + yy - \frac{ccxx}{3tt}} \times \frac{1}{2}px$. Q.E.D.

COROLLARY I.

When the generating hyperbola is equilateral, t is $= c$, and the rule becomes $\overline{YY + yy - \frac{1}{3}xx} \times \frac{1}{2}px$, which is the ſame with the rule for the fruſtum of the ſphere.

Co-

EXAMPLE.

If a caſk in the form of two fruſtums of an hyperboloid have its bung diameter 32 inches, its head diameter 24, and the diameter in the middle between the bung and head $\frac{8}{5}\sqrt{310}$; required the content in ale gallons, the length being 40 inches.

By problem 4 we ſhall have

$$\frac{10\sqrt{\overline{-12^2+\frac{64}{25}\times 310-16^2}\Big|^2-4\times 12^2\times 16^2}}{12^2-\frac{32}{25}\times 310+16^2}=$$

$$\frac{10\sqrt{\overline{\frac{64\times 310}{25}-20^2}\Big|^2-2^2\times 12^2\times 16^2}}{20^2-\frac{32\times 310}{25}}=\frac{10\sqrt{\overline{248-125}\big|^2-120^2}}{125-124}$$

$=30\sqrt{41^2-40^2}=30\sqrt{81}=270=$ the tranſverſe axe t. And $\frac{270}{10}\sqrt{\frac{12^2}{2}-\frac{16\times 310}{25}+\frac{16^2}{2}}=27\times 4\sqrt{\frac{3^2}{2}-\frac{62}{5}+\frac{4^2}{2}}$ $=108\sqrt{\frac{5^2}{2}-\frac{62}{5}}=108\sqrt{\frac{125-124}{10}}=108\sqrt{\frac{1}{10}}=$ the conjugate axe c.

Then

COROLLARY II.

When the axe is $=0$, or the hyperboloid becomes a cone; ſince then $Y:y::A+x:A$, or $Y-y:y::x:A=\frac{xy}{Y-y}$, and $t:c::A:y$, we ſhall have $\frac{cc}{tt}=\frac{yy}{AA}=\frac{\overline{Y-y}\big|^2}{xx}$, and therefore $\frac{ccxx}{3tt}=\frac{\overline{Y-y}\big|^2}{3}$; which being ſubſtituted for it in the rule $\overline{YY+yy-\frac{ccxx}{3tt}}\times\frac{1}{2}px$, produces $\overline{YY+Yy+yy}\times\frac{1}{3}px$ for the fruſtum of the cone, as it ought.

Then $\overline{DD+dd-\frac{aacc}{3tt}}\times\frac{1}{2}ap\times2=\overline{16^2+12^2-\frac{20^2\times108^2}{30\times270^2}}\times20p=\overline{20^2-\frac{20^2\times2^2}{30\times5^2}}\times20p=\frac{30\times25-4}{3}\times4^2\times2p=\frac{746\times32}{3}\times p=7957\frac{1}{3}\times3\cdot14159$ &c. $=24998\cdot69994216$ inches $=88\cdot64787213$ ale gallons.

RULE II.

Add together the ſquares of the greateſt and leaſt ſemi-diameters and the ſquare of the whole diameter in

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* DEMONSTRATION.

Put $x=$ the abſciſſa whoſe ordinate is δ, the axes being t and c; and, from the nature of the hyperbola, we ſhall have theſe three equations:

$$\begin{cases} tt\delta\delta=cc\times\overline{t+x}\times x=cc\times\overline{tx+xx}, \\ ttdd=cc\times\overline{t+x-\frac{1}{2}a}\times\overline{x-\frac{1}{2}a}=cc\times\overline{tx+xx-ax-\frac{1}{2}at+\frac{3}{4}aa}, \\ ttDD=cc\times\overline{t+x+\frac{1}{2}a}\times\overline{x+\frac{1}{2}a}=cc\times\overline{tx+xx+ax+\frac{1}{2}at+\frac{1}{4}aa}, \end{cases}$$

from the ſum of the two latter of which ſubtract the double of the former, and there will reſult $tt\times\overline{DD-2\delta\delta+dd}=\frac{1}{2}aacc$; and hence $\frac{aacc}{3tt}$ will be $=\frac{2DD-4\delta\delta+2dd}{3}$.

Which being ſubſtituted inſtead of it in the laſt rule will give $\frac{DD+4\delta\delta+dd}{6}\times ap$ for the content of the fruſtum required. *Q. E. D.*

COROLLARY I.

If d the leaſt diameter be ſuppoſed to become infinitely little or nothing, the above-rule will become $\frac{DD+4\delta\delta}{6}\times ap$ for the content of a ſegment.

COROLLARY II.

An hyperbolic fruſtum is equal to a cylinder of the ſame altitude and whoſe diameter is $\sqrt{\frac{DD+4\delta\delta+dd}{6}}$. And the ſegment is equal to the cylinder whoſe diameter is $\sqrt{\frac{DD+4\delta\delta}{6}}$, the altitude being the ſame.

in the middle, multiply the ſum by the altitude, and the product by 3·14159 &c. and one-ſixth of the laſt product will be the content.

That is, $\frac{DD + 4\delta\delta + dd}{6} \times ap =$ the content, putting D, δ, and d for the greateſt, middle, and leaſt ſemi-diameters, $a =$ the altitude, and $p = 3·14159$ &c.

EXAMPLE.

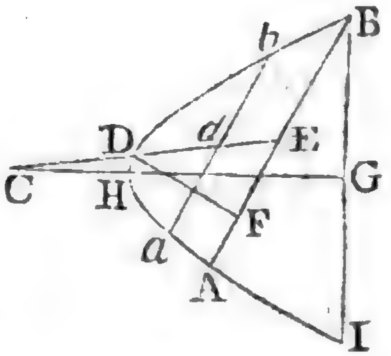
Taking here the ſame Example as before, we ſhall have $\frac{16^2 + \frac{64 \times 310}{25} + 12^2}{6} \times 40p = \frac{20^2 + \frac{64 \times 62}{5}}{6} \times 40p = \frac{5 \times 20^2 + 64 \times 62}{3} \times 4p = 7957\frac{1}{3}p =$ the content the ſame as before.

PROBLEM XII.

To find the content of the ſegment cut off an hyperboloid by a plane oblique to the axe of the ſolid.

RULE.

From the ſum of $2CD$ and DE ſubtract $\frac{1}{3}$ of DE, and divide the remainder by the double of the ſaid ſum; multiply the quotient by the product of the baſe and perpendicular altitude, and the laſt product will be the content.



That is, $\frac{2CD + DE - \frac{1}{3}DE}{4CD + 2DE} \times DF \times$ baſe = the content of ADB.——Where $2CD$ is the diameter to the double ordinate AB or tranſverſe diameter of the the baſe, DE the continuation of the diameter to the middle of AB, and DF perpendicular to AB.*

PRO-

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* DEMONSTRATION.

Put $2CD = a$, $DE = b$, $EB = c$, $Dd = x$, $db = y$, $B =$ the elliptic baſe whoſe tranſverſe axe is AB, and $s =$ the ſine of the angle E to the radius 1.

By the property of the hyperbola, $ab + bb : ax + xx :: cc : yy = cc \times \frac{ax + xx}{ab + bb}$; and becauſe parallel ſections are ſimilar, and ſimilar figures as the ſquares of their like diameters, we obtain $cc : yy :: B : B \times \frac{ax + xx}{ab + bb} =$ the ſection at ab.

Hence the fluxion of the ſolid is $Bs\dot{x} \times \frac{ax + xx}{ab + bb}$; whoſe fluent $Bsxx \times \frac{\frac{1}{2}a + \frac{1}{3}x}{ab + bb} = \frac{Bsxx}{2b} \times \frac{a + \frac{2}{3}x}{a + b}$ will be the content of aDb.

And when x or Dd becomes $= b$ or DE, the rule will be $\frac{1}{2}Bbs \times \frac{a + \frac{2}{3}b}{a + b} = \frac{1}{2}B \times DF \times \frac{a + \frac{2}{3}b}{a + b}$ for the ſolid ADB. Q.E.D.

COROLLARY I.

When E is a right angle, DE, DF and HG coincide, as do AB and BI; and the rule will become $\frac{1}{2}$ baſe $\times HG \times \frac{2CH + \frac{2}{3}HG}{2CH + HG}$ for the right ſegment, the ſame as in problem 10.

COROLLARY II.

If from $\frac{1}{2}$ baſe $BI \times HG \times \frac{2CH + \frac{2}{3}HG}{2CH + HG}$ the right ſegment IHB, be taken $\frac{1}{2}B \times DF \times \frac{2CD + \frac{2}{3}DE}{2CD + DE}$ the oblique ſegment ADB, there will remain the part BAI.

SCHOLIUM.

I might here proceed to find the ſlice parallel to the axe and the other parts dependent on it; but as there appears little occaſion for it, and it would run into a complex infinite ſeries, I have omitted it.

PROBLEM XIII.

* *To find the content of an hyperbolic ſpindle.*

RULE I.

Proceed as in rule 1 for the elliptic ſpindle, and the content will be obtained.

That

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* GENERAL INVESTIGATION.

Put $x = DE$, and $y = EF$; then

By the property of the curve, $c : t :: \sqrt{cc+xx} : \frac{t\sqrt{cc+xx}}{c} = FI$; hence $y = DG - IF = C - \frac{t\sqrt{cc+xx}}{c}$, and the fluxion of the ſolid will be $py y\dot{x} = p\dot{x} \times \overline{CC - \frac{2Ct\sqrt{cc+xx}}{c} + tt + \frac{ttxx}{cc}} = p\dot{x} \times \overline{tt - CC + \frac{ttxx}{cc} + 2C \times C - \frac{t\sqrt{cc+xx}}{c}} = p\dot{x} \times \overline{tt - CC + \frac{ttxx}{cc}} + 2Cy = 2pCy\dot{x} - \frac{ptt\dot{x}}{cc} \times \overline{\tfrac{1}{4}LL - xx}$, whoſe fluent gives $2pC \times$ area $EFBD - \frac{pttx}{cc} \times \overline{\tfrac{1}{4}LL - \tfrac{1}{3}xx} = 2pC \times$ area $EFBD - \frac{pttx}{cc} \times \frac{3LL - 4xx}{12}$ for the content of the fruſtum $BFML$. Or $2pC \times$ area $EFBNO - \frac{pttl}{cc} \times \frac{3LL - ll}{12} =$ the content of the middle zone $FBNPLM$, putting $l = 2x = EO$ its length.

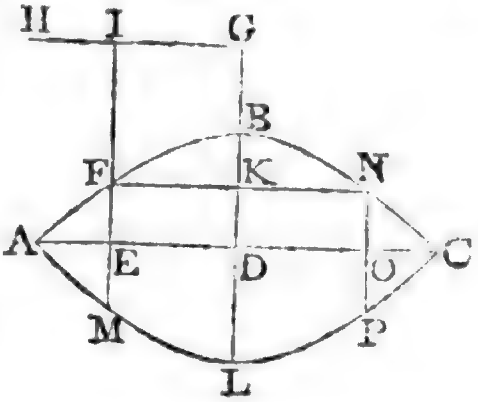
COROLLARY I.

When l becomes $= L$, the laſt theorem becomes $2pCS - \frac{pt^2L^3}{6cc}$ for the whole ſpindle $ABCLA$.

COROLLARY II.

If from the ſemi-ſpindle be ſubtracted the fruſtum, there will remain $2pC \times$ area $AFE - \frac{paatt}{cc} \times \frac{3L - 2a}{6}$ for the ſegment MAF, putting a for AE the altitude.

Co-



That is, $\overline{2CS - \frac{t^2L^3}{6cc}} \times p$ = the content of the ſpindle *ABCLA*; where *C* is $= GD$ the central diſtance or the diſtance of the centers of the ſpindle and generating area *ABC*, $S =$ the generating area *ABC*, $t = BG$ the ſemi-tranſverſe, $c = GH$ the ſemi-conjugate, $L = AC$ the length of the ſpindle, and $p = 3{\cdot}14159$ &c. As in corollary 1.

NOTE

COROLLARY III.

When the generating hyperbola is equilateral, we ſhall have $LL = 4CC - 4tt = 4CC - \overline{2C-D}|^2 = 4CD - DD$, and hence $C = \frac{LL+DD}{4D}$, which being ſubſtituted in the foregoing rules, we obtain $\frac{LL+DD}{2D} \times pS - \frac{1}{6}pL^3$ for the whole ſpindle, $\frac{LL+DD}{2D} \times p \times$ area $EFBNO - pl \times \frac{3LL-LL}{12}$ for the middle zone *FBNPLM*, and $\frac{LL+DD}{2D} \times p \times$ area $AFE - paa \times \frac{3L-2a}{6}$ for the ſegment *MAF*.

COROLLARY IV.

Putting *d* for *MF* the leaſt diameter of the fruſtum or zone, $n = \frac{1}{4}p = {\cdot}785398$ &c. and the reſt of the quantities as above; ſince LL is $= ll \times \frac{DC - \frac{1}{4}DD}{DC - dC - \frac{1}{4}DD + \frac{1}{4}dd}$, $\frac{tt}{cc} = \frac{DC - dC - \frac{1}{4}DD + \frac{1}{4}dd}{\frac{1}{4}ll}$, and the area $EFBNO = \frac{1}{2}dl + s$, putting *s* for the area *FBN*; if theſe values be ſubſtituted in the general forms, we ſhall obtain $\frac{1}{5}nl \times \overline{2DD + dd + 8C \times \overline{-D + d + \frac{3s}{l}}}$ for the value of the middle zone *FBNPLM*, and $\frac{1}{5}nL \times \overline{2DD + 8C \times \overline{-D + \frac{3S}{L}}}$ for that of the whole ſpindle.

Co-

NOTE.

If the generating hyperbola be equilateral, the theorem will be barely $\overline{\frac{LL+DD}{D}\times S-\frac{1}{3}L^3}\times\frac{1}{2}p$ for the content of the ſpindle, putting $D = BL$ the greateſt diameter. As in corollary 3.

RULE

COROLLARY V.

And if here, again, the generating hyperbola be ſuppoſed to be equilateral, ſince in that caſe ll is $= \overline{2C-d}|^2 - 4tt = \overline{2C-d}|^2 - \overline{2C-D}|^2 = -4Cd+dd+4CD-DD$, and hence $C = \frac{ll+DD-dd}{4D-4d} = \frac{LL+DD}{4D}$; we ſhall then have $\frac{1}{3}nL \times \overline{2DD+\frac{2LL+2DD}{D}\times\overline{-D+\frac{3S}{L}}}$ or $\frac{1}{6}pL\times\overline{3S\times\frac{LL+DD}{LD}-LL}$ for the whole ſpindle, and

$\frac{1}{3}nl\times\overline{2DD+dd+\frac{2ll+2DD-2dd}{D-d}\times\overline{-D+d+\frac{3s}{l}}}$

or $\frac{1}{6}pl\times\overline{\frac{1}{2}dd-ll+\frac{3s}{l}\times\frac{ll+DD-dd}{D-d}}$ for the zone.

Hence it may be obſerved that the content of an equilateral-hyperbolic-ſpindular caſk may be found from having only its length with the bung and head diameters given.

COROLLARY VI.

The fluent of $pyy\dot{x}$ or $p\dot{x}\times\overline{CC+tt+\frac{ttxx}{cc}-\frac{2Ct\sqrt{cc+xx}}{c}}$ is alſo $= px\times\overline{CC+tt+\frac{ttxx}{3cc}}-\frac{Ctx\sqrt{cc+xx}}{c}-Ctc\times$ hyp. log. of $\frac{x+\sqrt{cc+xx}}{c}$, which is another expreſſion of the content of the fruſtum $FBLM$.

COROLLARY VII.

Putting m for the diameter in the middle of the fruſtum or ſemi-ſpindle; then, by the nature of the hyperbola, we ſhall have

RULE II.

Divide 3 times the generating area by the length of the ſpindle, from the quotient ſubtract the greateſt diameter of the ſpindle; multiply the remainder by four times the central diſtance, and to the product add the ſquare of the greateſt diameter; then the ſum multiplied by the length, and the product by 3·14159 &c. $\frac{1}{6}$ of the laſt product will be the content.

That

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have $\overline{C - \frac{1}{2}d}|^2 - \overline{C - \frac{1}{2}D}|^2 : \overline{C - \frac{1}{2}m}|^2 - \overline{C - \frac{1}{2}D}|^2 :: 4 : 1$, hence $\overline{C - \frac{1}{2}d}|^2 = 4 \times \overline{C - \frac{1}{2}m}|^2 - 3 \times \overline{C - \frac{1}{2}D}|^2$, and $C = \frac{1}{4} \times \frac{4m^2 - 3D^2 - d^2}{4m - 3D - d}$ in the fruſtum, or $C = \frac{1}{4} \times \frac{4m^2 - 3D^2}{4m - 3D}$ in the whole ſpindle; and hence the rules in corollary 4 will become $\overline{DD + \frac{4m^2 - 3D^2}{4m - 3D} \times \frac{3S}{L} - D} \times \frac{1}{6}pL$ for the whole ſpindle; and $\overline{DD + \frac{1}{2}dd + \frac{4m^2 - 3D^2 - d^2}{4m - 3D - d} \times \frac{3s}{l} + d - D} \times \frac{1}{6}pl$ for the zone..

COROLLARY VIII..

And if the generating hyperbola be equilateral, then $\frac{1}{4}ll = \overline{C - \frac{1}{2}d}|^2 - \overline{C - \frac{1}{2}D}|^2 =$ (by writing for C its value found in the laſt corollary) $\frac{\overline{\overline{m - d}|^2 - \frac{1}{4} \times \overline{D - d}|^2}|^2 - \overline{\overline{D - m}|^2 - \frac{1}{4} \times \overline{D - d}|^2}|^2}{\overline{4m - 3D - d}|^2}$;

hence $\frac{2\sqrt{\overline{\overline{m - d}|^2 - \frac{1}{4} \times \overline{D - d}|^2}|^2 - \overline{\overline{D - m}|^2 - \frac{1}{4} \times \overline{D - d}|^2}|^2}}{4m - 3D - d} = l$ the length of the zone; and, when d vaniſhes, we have

$\frac{2\sqrt{\overline{m^2 - \frac{1}{4}D^2}|^2 - \overline{\overline{D - m}|^2 - \frac{1}{4}D^2}|^2}}{4m - 3D} = L$ the length of the whole ſpindle. So that when the hyperbola is equilateral, the length of the ſpindle need not be given if the diameters D and m be given.

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That is, $\overline{DD + 4C \times \overline{\frac{3S}{L} - D}} \times \frac{1}{8}pL$ = the content; the letters as before.——By corollary 4.

NOTE.

If the generating hyperbola be ſuppoſed to be equilateral, 4 times the central diſtance will be $= \frac{LL + DD}{D}$, and then $\overline{3S \times \frac{LL + DD}{LD} - LL} \times \frac{1}{6}pL =$ the content.——By corollary 5.

RULE III.

Divide the difference between 4 times the ſquare of the middle diameter and 3 times the ſquare of the greateſt, by the difference between 4 times the ſaid middle diameter and 3 times the greateſt, and $\frac{1}{4}$ of the quotient will be the central diſtance.

Then proceed as in the laſt rule.

That is, $\overline{DD + \frac{4mm - 3DD}{4m - 3D} \times \overline{\frac{3S}{L} - D}} \times \frac{1}{8}pL$ = the content.——By corollary 7.

NOTE.

When the generating hyperbola is equilateral, the content may be found without having the length L given, for then L is =

$$\frac{2\sqrt{\overline{mm - \frac{1}{4}DD}|^2 - \overline{\overline{D - m}|^2 - \frac{1}{4}DD}|^2}}{4m - 3D},$$ by corollary 8.

Another rule might be drawn from corollary 6.

PRO-

PROBLEM XIV.

To find the content of a zone or double fruſtum of an hyperbolic ſpindle.

RULE I.

Uſe here rule 1 for the zone of an elliptic ſpindle, and the content will be obtained.

That is, $2pC \times$ area $EFBNO - \frac{pttl}{cc} \times \frac{3LL - ll}{12} =$ the content of the zone $FBNPLM$; where $p = 3{\cdot}14159$ &c. $C = GD$ the central diſtance, $t = BG$ the ſemi-tranſverſe axe, $c = GH$ the ſemi-conjugate, $l = EO$ the length of the zone, and $L = AC$ the length of the whole ſpindle.——By the general inveſtigation of the laſt problem.

NOTE.

When the generating hyperbola is equilateral, the rule will be barely $p \times \frac{LL + DD}{2D} \times$ area $EFBNO - pl \times \frac{3LL - ll}{12}$.——By corollary 3 to the laſt problem. Putting $D = BL$ the greateſt diameter.

RULE II.

Uſe the ſecond rule for the zone of an elliptic ſpindle, only here *add* the firſt general product inſtead of ſubtracting it, and the content will be got.

That is, $\overline{2DD + dd + 8C \times \frac{3s}{l} + d - D} \times \frac{1}{12}pl = FBNPLM$. Where $d = FM$ the leaſt diameter,

s = the area FBN, and the other letters as before.——By corollary 4 to the laſt problem.

NOTE.

If the generating hyperbola be equilateral, 4 times the central diſtance will be $= \frac{ll + DD + dd}{D - d}$, and then the content of the zone will be $\overline{\tfrac{3}{2}dd - ll + \tfrac{3s}{l} \times \frac{ll + DD - dd}{D - d}} \times \tfrac{1}{6}pl$.——By corollary 5 to the laſt problem.

RULE III.

From 4 times the ſquare of the diameter equidiſtant from the greateſt and leaſt ſubtract the ſum of the ſquare of the leaſt and 3 times the ſquare of the greateſt diameter, and from 4 times the ſaid middle diameter take the ſum of the leaſt and 3 times the greateſt; then divide the former difference by the latter, and $\frac{1}{4}$ of the quotient will be the central diſtance.

That is, $\frac{1}{4} \times \frac{4m^2 - 3D^2 - d^2}{4m - 3D - d} = C$, m being the middle diameter.—By corollary 7 to the laſt problem.

Then proceed as in the laſt rule.

NOTE.

When the generating hyperbola is equilateral, l is $= \frac{2\sqrt{\overline{\overline{m-d}|^2 - \frac{1}{3} \times \overline{D-d}|^2}|^2 - \overline{\overline{D-m}|^2 - \frac{1}{4} \times \overline{D-d}|^2}|^2}}{4m - 3D - d}$ by corollary 8 to the laſt problem; ſo that the content may then be found without having the length given.

Another rule might be drawn from corollary 6 to the laſt problem.

PROBLEM XV.

To find the content of the ſegment of an hyperbolic ſpindle.

RULE.

Uſe here the rule for the ſegment of an elliptic ſpindle, only inſtead of what is there called the leſs axe take here the tranſverſe, and inſtead of the greater the conjugate axe; and the content will be obtained.——By corollary 2 to the 13th problem.

That is, $2pC \times \text{area } AFE - \frac{paatt}{cc} \times \frac{3L - 2a}{6} =$ the ſegment MAF. Where a is $= AE$ its altitude, and the other letters as in the laſt problems.

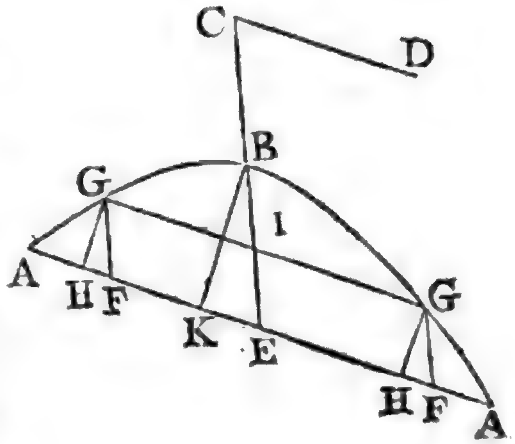
Other rules might be found by ſubtracting the fruſtum from the ſemi-ſpindle according to the correſponding rules of the laſt two problems.

PROBLEM XVI.

To find the content of an univerſal hyperbolic ſpindle; that is, of a ſolid generated from the revolution of an hyperbola about an ordinate to any diameter.

INVESTIGATION.

Put $a = BC$ the ſemi-diameter to which belongs the double ordinate AA about which the figure ABA revolves, $c = CD$ its ſemi-conjugate, $b = AE$, $C = EC$, $z = EF$, $y = FG$ parallel to CE, $m =$ ſine of



of the angle E or F. and $n =$ its coſine to the radius 1, alſo $p = 3{\cdot}14159$ &c. and let GH be perpendicular to AE.

Then, by the nature of the hyperbola, $c : a :: \sqrt{cc + zz} : \frac{a\sqrt{cc+zz}}{c} = CI$; hence $y = C - CI = C - \frac{a\sqrt{cc+zz}}{c}$: but GH is $= my$, and $HF = ny$; hence EH is $= z \pm ny$, and the fluxion of the ſolid is $p \times GH^2 \times \dot{HK} = p \times GH^2 \times \dot{HE} = pmmyy \times \overline{\dot{z} \pm n\dot{y}} = pmm \times : \pm ny^2\dot{y} + C^2\dot{z} - \frac{2Ca\dot{z}\sqrt{cc+zz}}{c} + a^2\dot{z} + \frac{a^2z^2\dot{z}}{cc}$; whoſe correct fluent is $pmm \times : \mp \frac{1}{3}ny^3 + C^2z + a^2z + \frac{a^2z^3}{3c^2} - \frac{Caz\sqrt{cc+zz}}{c} - Cac \times$ hyp. log. of $\frac{z+\sqrt{cc+zz}}{c} =$ the ſolid generated by $KBGH$.

And when z becomes $= b$, the above fluent becomes $pmm \times : \mp C^2b + a^2b + \frac{a^2b^3}{3c^2} - \frac{Cab\sqrt{cc+bb}}{c} - Cac \times$ hyp. log. of $\frac{b+\sqrt{cc+bb}}{c}$ for half the content of the whole ſolid ABA.

The ſolution of this problem was never but once before attempted, viz. in a periodical performance, but the ſolution there given is wrong, for the fluxion and fluent of the general fruſtum was there falſely aſſigned.

SEC-

SECTION VI.

Of the regular bodies.

A Regular ſolid or body, is a ſolid contained under ſome number of like, equal, and regular plane figures.

The plane figures, under which the ſolid is contained, are the faces of the ſolid. And the ſides of the plane figures are the edges or linear ſides of the ſolid.

There are only five ſorts of regular ſolids, viz. the tetraedron, or regular triangular pyramid, having four triangular faces; the hexaedron, or cube, having ſix ſquare faces; the octaedron, having eight triangular faces; the dodecaedron, having twelve pentagonal faces; and the icoſaedron, having twenty triangular faces.

PROBLEM I.

To deſcribe or form the regular ſolids.

RULE.

Having deſcribed figures, as below, on paſteboard or ſome other pliable matter, cut them out by the extreme ſides, and cut the other lines half through, then fold them at theſe lines ſo cut, till the ſides meet, which being faſtened together with glue, &c. you will have the form of the bodies. Viz. figure 1 will form the tetraedron; figure 2, the hexaedron; figure 3, the octaedron; figure 4, the dodecaedron; and figure 5, the icoſaedron.

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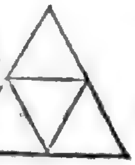

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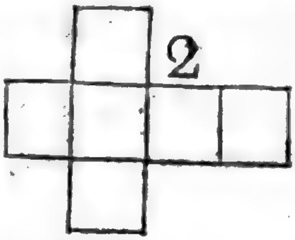

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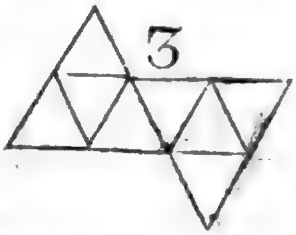

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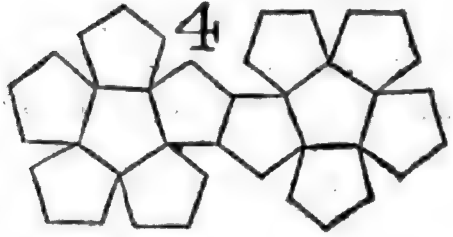

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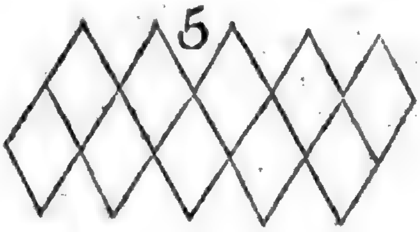

PROBLEM II.

To find the ſurface of a tetraedron.

RULE.

Multiply the ſquare of the linear edge by the root of 3, and the product will be the whole ſurface, or the ſum of the four faces.

That is, $A^2\sqrt{3}$ = the whole ſurface; A being the linear edge, or a ſide of one of the faces.*

EXAMPLE.

If each ſide of a face of a tetraedron be 1, required the whole ſurface.

Here $A = 1$; and therefore $A^2\sqrt{3} = \sqrt{3} = 1\cdot7320508$ = the ſurface required.

PRO-

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* DEMONSTRATION.

For, by rule 2 prob. 4 ſect. 1 part 2, one of the faces will be $= \frac{1}{4}\sqrt{3} \times A^2$; which being multiplied by 4, gives $A^2\sqrt{3}$ for all the four faces. *Q.E.D.*

PROBLEM III.

To find the ſolidity of a tetraedron.

*RULE.

Multiply $\frac{1}{12}$ of the cube of the linear ſide by the root

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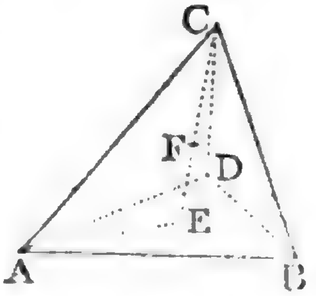
* DEMONSTRATION.



From one angle C of the tetraedron let fall a perpendicular CE upon the oppoſite ſide ADB, and draw EA.

Then $AC^2 = CE^2 + EA^2$; but, from the tables in pages 81 and 84, it appears that $\frac{1}{3}AC^2 = \frac{1}{3}AB^2$ is $= AE^2$; wherefore, by ſubtracting this equation from the former, $\frac{2}{3}AC^2$ will be $= CE^2$, and hence $CE = AC\sqrt{\frac{2}{3}}$; but ABD is $= \frac{1}{4}AB^2\sqrt{3} = \frac{1}{4}AC^2\sqrt{3}$: Then $\frac{1}{3}CE \times ABD = \frac{1}{3}AC\sqrt{\frac{2}{3}} \times \frac{1}{4}AC^2\sqrt{3} = \frac{1}{12}AC^3\sqrt{2} =$ the ſolidity. *Q.E.D.*

COROLLARY I.

A cube is to a tetraedron of the ſame linear ſide, as 1 is to $\frac{1}{12}\sqrt{2}$, or as 12 is to $\sqrt{2}$.

COROLLARY II.

Put r for the radius EF of the incribed ſphere, A for the linear ſide AC, B for the whole ſurface, and C for the ſolidity; and we ſhall have $r = \frac{3C}{B} = \left(\frac{\frac{1}{4}A^3\sqrt{2}}{A^2\sqrt{3}} = \frac{1}{4}A\sqrt{\frac{2}{3}} =\right) \frac{1}{12}A\sqrt{6}$.

COROLLARY III.

Hence alſo r or FE is $= \frac{1}{4}EC$; for, by the demonſtration, CE is $= A\sqrt{\frac{2}{3}}$.

COROLLARY IV.

And hence the radius of the circumſcribed ſphere is equal to 3 times that of the inſcribed; that is, $R = FC = CE - EF = 4FE - EF = 3FE = 3r = \frac{1}{4}A\sqrt{6}$.

SCHOLIUM.

From this problem and its corollaries, together with the former problem, are deduced theſe equations.

1. $A = \sqrt{\frac{1}{3}B\sqrt{3}} = \sqrt[3]{6C\sqrt{2}} = 2r\sqrt{6} = \frac{2}{3}R\sqrt{6}$.
2. $B = A^2\sqrt{3} = 6\sqrt[3]{C^2\sqrt{3}} = 24r^2\sqrt{3} = \frac{8}{3}R^2\sqrt{3}$.
3. $C = \frac{1}{12}A^3\sqrt{2} = \frac{1}{36}B\sqrt{2B\sqrt{3}} = 8r^3\sqrt{3} = \frac{8}{27}R^3\sqrt{3}$.
4. $r = \frac{1}{12}A\sqrt{6} = \frac{1}{12}\sqrt{2B\sqrt{3}} = \frac{1}{2}\sqrt[3]{\frac{1}{3}C\sqrt{3}} = \frac{1}{3}R$.
5. $R = \frac{1}{4}A\sqrt{6} = \frac{1}{4}\sqrt{2B\sqrt{3}} = \frac{3}{2}\sqrt[3]{\frac{1}{3}C\sqrt{3}} = 3r$.

root of 2, and the product will be the ſolidity of the tetraedron required.

That is, $\frac{1}{12}A^3\sqrt{2}$ = the ſolidity.

EXAMPLE.

If the linear ſide of a tetraedron be 1; required the ſolidity.

Here $A = 1$, and $\frac{1}{12}A^3\sqrt{2} = \frac{1}{12}\sqrt{2} = \cdot 11785113$ = the ſolidity required.

PROBLEM IV.

To find the ſurface or ſolidity of a hexaedron or cube.

RULE.

It is evident that 6 times the ſquare of the linear ſide will be equal to the whole ſurface, and the cube of the linear ſide equal to the ſolidity of the hexaedron or cube.[*]

PRO-

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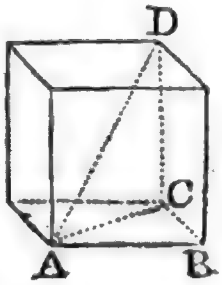
[*] SCHOLIUM.

Since it is evident that the diameter of the inſcribed ſphere is equal to the linear ſide of the hexaedron, and that the diameter of the circumſcribed ſphere is $AD = \sqrt{DC^2 + CA^2} = \sqrt{DC^2 + CB^2 + BA^2} = \sqrt{3 \cdot AB^2} = AB\sqrt{3}$; if A, B, and C be put to denote the linear ſide, the ſurface, and the ſolidity of a hexaedron, and r, R for the radius of the inſcribed and circumſcribed ſphere; then in the hexaedron or cube we ſhall have theſe equations.



1. $A = \sqrt{\tfrac{1}{6}B} = \sqrt[3]{C} = 2r = \tfrac{2}{3}R\sqrt{3}$.
2. $B = 6A^2 = 6\sqrt[3]{C^2} = 24r^2 = 8R^2$.
3. $C = A^3 = \tfrac{1}{6}B\sqrt{\tfrac{1}{6}B} = 8r^3 = \tfrac{8}{9}R^3\sqrt{3}$.
4. $r = \tfrac{1}{2}A = \tfrac{1}{2}\sqrt{\tfrac{1}{6}B} = \tfrac{1}{2}\sqrt[3]{C} = \tfrac{1}{3}R\sqrt{3}$.
5. $R = \tfrac{1}{2}A\sqrt{3} = \tfrac{1}{2}\sqrt{\tfrac{1}{2}B} = \tfrac{1}{2}\sqrt{3} \times \sqrt[3]{C} = r\sqrt{3}$.

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PROBLEM V.

To find the ſurface of an octaedron.

RULE.

Multiply 2 times the ſquare of the linear ſide by the root of 3, and the product will be the ſurface required.*

That is, $2A^2\sqrt{3}$ = the ſurface; A being the linear ſide.

EXAMPLE.

The linear ſide of an octaedron being 1, it is required to find the ſolidity.

Here $A = 1$, and $2A^2\sqrt{3} = 2\sqrt{3} = 3{\cdot}4641016 =$ the ſurface required.

PROBLEM VI.

To find the ſolidity of an octaedron.

RULE.

Multiply $\frac{1}{3}$ of the cube of the linear ſide by the root of 2, and the product will be the content of the octaedron.

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* DEMONSTRATION.

By rule 2 prob. 4 ſect. 1 part 2, one face is $= \frac{1}{4}A^2\sqrt{3}$; conſequently 8 faces, or the whole ſurface, will be $= 2A^2\sqrt{3}$. *Q.E.D.*

That is, $\frac{1}{3}A^3\sqrt{2} = C =$ the content; A being the linear ſide.

Ex-

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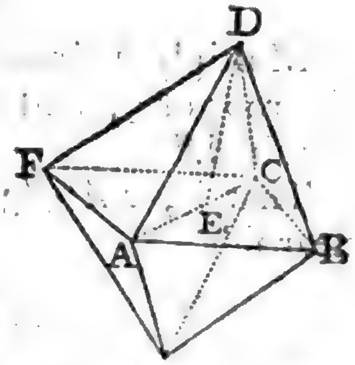
* DEMONSTRATION.



Let E be the middle of the diagonal AC of the ſolid, or the center of the ſolid; and draw DE which will be equal to EA.

Then, ſince the ſolid is evidently compoſed of two equal ſquare pyramids, each baſe of which $ABCF$ is equal to the ſquare of the linear ſide of the ſolid, and the altitude of each equal to DE or EA half the diagonal of that ſquare; we ſhall have $ABCF \times \frac{2}{3}AE = AB^2 \times \frac{1}{3}AC = \frac{1}{3}AB^2\sqrt{AB^2 + BC^2} = \frac{1}{3}AB^2\sqrt{2AB^2} = \frac{1}{3}AB^3\sqrt{2}$ for the ſolidity. Q.E.D.

COROLLARY I.

The radius AE of the circumſcribed ſphere is $= \frac{1}{2}AC = \frac{1}{2}\sqrt{2AB^2} = \frac{1}{2}AB\sqrt{2} = AB\sqrt{\frac{1}{2}}$.

COROLLARY II.

The radius of the inſcribed ſphere is equal to the quotient ariſing from the diviſion of 3 times the ſolidity by the whole ſurface = (from this problem and the laſt) $AB^3\sqrt{2} \div 2AB^2\sqrt{3} = \frac{1}{2}AB\sqrt{\frac{2}{3}} = \frac{1}{6}AB\sqrt{6} = AB\sqrt{\frac{1}{6}}$.

SCHOLIUM.

From the two laſt problems and the corollaries are deduced the following equations; in which A, B, C, r, R, denote the linear ſide of an octaedron, B the whole ſurface, C the ſolidity, r the radius of the inſcribed ſphere, and R the radius of the circumſcribed ſphere.

1. $A = \sqrt{\frac{B\sqrt{3}}{6}} = \sqrt[3]{\frac{3C\sqrt{2}}{2}} = r\sqrt{6} = R\sqrt{2}$.

2. $B = 2A^2\sqrt{3} = 6\sqrt[3]{\frac{C^2\sqrt{3}}{2}} = 12r^2\sqrt{3} = 4R^2\sqrt{3}$.

3. $C = \frac{1}{3}A^3\sqrt{2} = \frac{B\sqrt{B\sqrt{3}}}{18} = 4r^3\sqrt{3} = \frac{4}{3}R^3$.

4. $r = \frac{1}{6}A\sqrt{6} = \frac{1}{6}\sqrt{B\sqrt{3}} = \sqrt[3]{\frac{C\sqrt{3}}{12}} = \frac{1}{3}R\sqrt{3}$.

5. $R = \frac{1}{2}A\sqrt{2} = \sqrt{\frac{B\sqrt{3}}{12}} = \sqrt[3]{\frac{3}{4}C} = r\sqrt{3}$.

EXAMPLE.

If the linear ſide of an octaedron be 1, what is its ſolidity.

Here $A = 1$, and therefore $\frac{1}{3}A^3\sqrt{2} = \frac{1}{3}\sqrt{2} = \cdot 47140452079 =$ the content required.

PROBLEM VII.

To find the ſurface of a dodecaedron.

RULE.

To 5 add 2 times the root of 5, and divide the ſum by 5; multiply the root of the quotient by 15 times the ſquare of the linear ſide, and the product will be the ſurface of the dodecaedron.

That is, $15A^2\sqrt{\frac{5+2\sqrt{5}}{5}} =$ the ſurface; A being the linear ſide.*

EXAMPLE.

If the linear ſide be 1, required the ſurface.

Here $A = 1$, and therefore $15A^2\sqrt{\frac{5+2\sqrt{5}}{5}} = 15\sqrt{\frac{5+2\sqrt{5}}{5}} = 20\cdot 6457788075 =$ the ſurface required.

PRO-

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* DEMONSTRATION.

By rule 2 prob. 4 ſect. 1 part 2, the area of one face is $\frac{5}{4}A^2\sqrt{1+\frac{2}{5}\sqrt{5}}$, and therefore 12 faces or the whole ſurface will be $12 \times \frac{5}{4}A^2\sqrt{1+\frac{2}{5}\sqrt{5}} = 15A^2\sqrt{\frac{5+2\sqrt{5}}{5}}$. *Q.E.D.*

PROBLEM VIII.

To find the ſolidity of a dodecaedron.

RULE.

To 21 times the root of 5 add 47, and divide the ſum by 40; multiply the root of the quotient by

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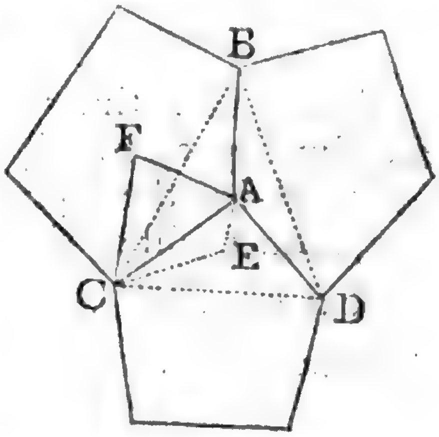
* DEMONSTRATION.



Let A be a ſolid angle of the dodecaedron, and let the extremities of the ſides AB, AC, AD, of the faces which form the angle be connected by the right lines BC, CD, DB, forming the equilateral triangle BCD within the ſolid; upon the center E of which triangle let fall the perpendicular AE, and to the center F of one of the faces draw the lines AF, CF.

The angle CAD contains 108 degrees, whoſe ſine is $\frac{1}{4}\sqrt{10+2\sqrt{5}}$, to the radius 1; and the angle ADC contains 36 degrees, whoſe ſine is $\frac{1}{4}\sqrt{10-2\sqrt{5}}$; wherefore as $\sqrt{10-2\sqrt{5}} : \sqrt{10+2\sqrt{5}} :: AC : CD = AC\sqrt{\frac{10+2\sqrt{5}}{10-2\sqrt{5}}} = AC\sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}} = AC\sqrt{\frac{\overline{5+\sqrt{5}}|^{2}}{25-5}} = \frac{5+\sqrt{5}}{2\sqrt{5}}AC = \frac{1+\sqrt{5}}{2}AC$: And much after the ſame manner, we find $EC = \frac{1}{3}CD\sqrt{3} = CD\sqrt{\frac{1}{3}} = \frac{1+\sqrt{5}}{2\sqrt{3}}AC$. Hence $EA = \sqrt{AC^2 - CE^2} = \sqrt{AC^2 - \overline{\frac{1+\sqrt{5}}{2\sqrt{3}}}\Big|^{2} AC^2} = AC\sqrt{1-\frac{3+\sqrt{5}}{6}} = AC\sqrt{\frac{3-\sqrt{5}}{6}}$. Then becauſe the chord of an arc is a mean proportional between its verſed ſine and the diameter, and AE is the verſed ſine of the arc whoſe chord is AC and diameter equal to that of by 5 times the cube of the linear ſide, and the product will be the ſolidity.

That is, $5A^3\sqrt{\frac{47+21\sqrt{5}}{40}}$ = the ſolidity; A being the linear ſide.

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of the circumſcribed ſphere; we ſhall have $\frac{CA^2}{2AE} = \frac{CA^2}{2AC\sqrt{\frac{3-\sqrt{5}}{6}}}$ $= \frac{1}{2}AC\sqrt{\frac{6}{3-\sqrt{5}}} = \frac{1}{2}AC\sqrt{6\times\frac{3+\sqrt{5}}{9-5}} = \frac{1}{2}AC\sqrt{3\times\frac{6+2\sqrt{5}}{4}}$ $= AC\times\frac{1+\sqrt{5}}{4}\sqrt{3} = \frac{\sqrt{3}+\sqrt{15}}{4}AC = R$ the radius of the circumſcribed ſphere.

Again, the angle AFC contains 72 degrees, whoſe ſine is $\frac{1}{4}\sqrt{10+2\sqrt{5}}$; and the angle ACF 54 degrees, whoſe ſine is $\frac{1+\sqrt{5}}{4}$; wherefore as $\sqrt{10+2\sqrt{5}} : 1+\sqrt{5} :: CA : AF =$ $\frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}}AC = AC\sqrt{\frac{5+\sqrt{5}}{10}}$.

But, ſince the radius of the circumſcribed ſphere is the hypothenuſe of a right-angled triangle whoſe two legs are AF and the radius of the inſcribed ſphere, we ſhall have $\sqrt{R^2 - AF^2} =$ $\sqrt{\left|\frac{\sqrt{3}+\sqrt{15}}{4}\right|^2 AC^2 - \frac{5+\sqrt{5}}{10}AC^2} = AC\sqrt{\frac{18+6\sqrt{5}}{16} - \frac{5+\sqrt{5}}{10}}$ $= AC\sqrt{\frac{25+11\sqrt{5}}{40}} = r$ the radius of the inſcribed ſphere.

Then, becauſe the ſolidity of any regular ſolid is equal to the ſurface drawn into $\frac{1}{3}$ of the radius of the inſcribed ſphere, and the ſurface B equal to $15A^2\sqrt{\frac{5+2\sqrt{5}}{5}}$ by the laſt problem, we ſhall have $B\times\frac{1}{3}r = 15A^2\sqrt{\frac{5+2\sqrt{5}}{5}}\times\frac{1}{3}A\sqrt{\frac{25+11\sqrt{5}}{40}} =$ $5A^3\sqrt{\frac{47+21\sqrt{5}}{40}} = C$ the ſolidity. *Q.E.D.*

EXAMPLE.

If the linear ſide be 1, required the ſolidity of the dodecaedron.

Here $A = 1$, and therefore $5A^3\sqrt{\frac{47 + 21\sqrt{5}}{40}} = 5\sqrt{\frac{47 + 21\sqrt{5}}{40}} = 7{\cdot}66311896 05$ = the content required.

PRO-

COROLLARY.

From the demonſtration it appears that the radius of the circumſcribed ſphere is $R = \frac{\sqrt{3} + \sqrt{15}}{4}A$, and that of the inſcribed ſphere is $r = A\sqrt{\frac{25 + 11\sqrt{5}}{40}}$.

SCHOLIUM.

Putting, as before, A, B, C for the linear ſide, ſurface, and ſolidity, and R, r for the radius of the circumſcribed and inſcribed ſphere reſpectively; we ſhall have in the dodecaedron

$$1.\ A = \sqrt{\frac{B\sqrt{5 - 2\sqrt{5}}}{15}} = \sqrt[3]{\frac{C\sqrt{470 - 210\sqrt{5}}}{5}} = r\sqrt{50 - 22\sqrt{5}} = \frac{\sqrt{15} - \sqrt{3}}{3}R.$$

$$2.\ B = 15A^2\sqrt{\frac{5 + 2\sqrt{5}}{5}} = 3\sqrt[3]{10C^2\sqrt{130 - 58\sqrt{5}}} = 30r^2\sqrt{130 - 58\sqrt{5}} = 10R^2\sqrt{\frac{10 - 2\sqrt{5}}{5}}.$$

$$3.\ C = 5A^3\sqrt{\frac{47 + 21\sqrt{5}}{40}} = \frac{B}{30}\sqrt{\frac{B\sqrt{650 + 290\sqrt{5}}}{6}} = 10r^3\sqrt{130 - 58\sqrt{5}} = \frac{20R^3}{3}\sqrt{\frac{3 + \sqrt{5}}{30}}.$$

$$4.\ r = \frac{A\sqrt{250 + 110\sqrt{5}}}{20} = \frac{1}{20}\sqrt{\frac{2B\sqrt{650 + 290\sqrt{5}}}{3}} = \sqrt[3]{\frac{C\sqrt{650 + 290\sqrt{5}}}{200}} = R\sqrt{\frac{5 + 2\sqrt{5}}{15}}.$$

$$5.\ R = \frac{\sqrt{15} + \sqrt{3}}{4}A = \sqrt{\frac{B\sqrt{10 + 2\sqrt{5}}}{40}} = \sqrt[3]{\frac{3C\sqrt{90 - 30\sqrt{5}}}{40}} = r\sqrt{15 - 6\sqrt{5}}.$$

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PROBLEM IX.

To find the ſurface of an icoſaedron.

RULE.

* Multitiply 5 times the ſquare of the linear ſide by the root of 3, and the product will be the ſurface.

That is, $5A^2\sqrt{3}$ = the ſurface; A being the linear ſide.

EXAMPLE.

The linear ſide of an icoſaedron being 1; required the ſurface.

Here $A = 1$, and therefore $5A^2\sqrt{3} = 5\sqrt{3} = 8{\cdot}660254035$.

PROBLEM X.

To find the ſolidity of an icoſaedron.

RULE.

To 7 add 3 times the root of 5, divide the ſum by 2, multiply the root of the quotient by five-ſixths of the cube of the linear ſide, and the product will be the content of the icoſaedron.*

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* DEMONSTRATION.

For one face is $= \frac{1}{4}A^2\sqrt{3}$, and therefore twenty faces or the whole ſurface will be $20 \times \frac{1}{4}A^2\sqrt{3} = 5A^2\sqrt{3}$. *Q.E.D.*

That is, $\frac{5}{6}A^3\sqrt{\frac{7+3\sqrt{5}}{2}}$ = the content; A being the linear ſide.

Ex-

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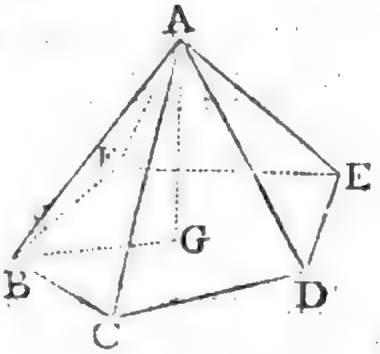
* DEMONSTRATION.



Let A be a ſolid angle, of the icoſaedron, formed by five faces or triangles whoſe baſes form the pentagon $BCDEF$, upon whoſe center G let fall the perpendicular AG, and draw GB.

In the demonſtration of problem 8 it was found that GB is $= BA\sqrt{\frac{5+\sqrt{5}}{10}}$, and that the radius ρ of the circle circumſcribing one face ABC is $= AB\sqrt{\frac{1}{3}}$. But the radius of the circumſcribed ſphere is $R = \frac{BA^2}{2AG} = \frac{BA^2}{2\sqrt{AB^2 - BG^2}} = \frac{BA^2}{2\sqrt{AB^2 - \frac{5+\sqrt{5}}{10}AB^2}} = \frac{BA}{2\sqrt{1 - \frac{5+\sqrt{5}}{10}}} = \frac{BA}{2\sqrt{\frac{5-\sqrt{5}}{10}}} = \frac{1}{2}AB\sqrt{\frac{10}{5-\sqrt{5}}} = \frac{1}{2}AB\sqrt{\frac{10}{5-\sqrt{5}} \times \frac{5+\sqrt{5}}{5+\sqrt{5}}} = \frac{1}{2}AB\sqrt{\frac{10 \times \overline{5+\sqrt{5}}}{25-5}} = AB\sqrt{\frac{5+\sqrt{5}}{8}}$; and R is the hypothenuſe of a right-angled triangle of which the one leg is the radius ρ of the circle circumſcribing the face, and the other the radius r of the inſcribed ſphere; wherefore r is $= \sqrt{R^2 - \rho^2} = \sqrt{\frac{5+\sqrt{5}}{8}AB^2 - \frac{1}{3}AB^2} = AB\sqrt{\frac{15+3\sqrt{5}-8}{8 \times 3}} = AB\sqrt{\frac{7+3\sqrt{5}}{24}}$. Conſequently the ſolidity $C = \frac{1}{3}rB$ (B being the whole ſurface) will be $\frac{1}{3}AB\sqrt{\frac{7+3\sqrt{5}}{24}} \times 5AB^2\sqrt{3} = \frac{5}{3}AB^3\sqrt{\frac{7+3\sqrt{5}}{8}} = \frac{5}{6}AB^3\sqrt{\frac{7+3\sqrt{5}}{2}}$.

Q.E.D.

SCHO-

EXAMPLE.

The linear ſide of an icoſaedron being 1, required the ſolidity.

Here $A = 1$, and therefore $\frac{5}{6}A^3\sqrt{\frac{7+3\sqrt{5}}{2}} = \frac{5}{6}\sqrt{\frac{7+3\sqrt{5}}{2}} = 2{\cdot}1816949905 =$ the content required.

SCHOLIUM.

From this problem and the laſt may be deduced the following equations in the icoſaedron, the letters being as before.

1. $A = \sqrt{\frac{B\sqrt{3}}{15}} = \sqrt[3]{\frac{6}{5}C\sqrt{\frac{7-3\sqrt{5}}{2}}} = r\sqrt{42-18\sqrt{5}} = R\sqrt{\frac{10-2\sqrt{5}}{5}}$.

2. $B = 5A^2\sqrt{3} = 3\sqrt[3]{70\sqrt{3}-30\sqrt{15}} = 3r^2 \times \overline{7\sqrt{3}-3\sqrt{15}} = 2R^2 \times \overline{5\sqrt{3}-\sqrt{15}}$.

3. $C = \frac{5}{6}A^3\sqrt{\frac{7+3\sqrt{5}}{2}} = \frac{B}{18}\sqrt{\frac{7\sqrt{3}+3\sqrt{15}}{10}B} = 10r^3 \times \overline{7\sqrt{3}-3\sqrt{15}} = \frac{2}{3}R^3\sqrt{10+2\sqrt{5}}$.

4. $r = \frac{1}{2}A\sqrt{\frac{7+3\sqrt{5}}{6}} = \frac{1}{6}\sqrt{\frac{7\sqrt{3}+3\sqrt{15}}{10}B} = \frac{1}{2}\sqrt[3]{\frac{7\sqrt{3}+3\sqrt{15}}{30}C} = R\sqrt{\frac{5+2\sqrt{5}}{15}}$.

5. $R = \frac{1}{2}A\sqrt{\frac{5+\sqrt{5}}{2}} = \frac{1}{2}\sqrt{\frac{5\sqrt{3}+\sqrt{15}}{30}B} = \sqrt[3]{\frac{1}{4}C\sqrt{\frac{5-\sqrt{5}}{10}}} = r\sqrt{15-6\sqrt{5}}$.

SECTION VII.

Of ſolid Rings.

DEFINITIONS.

BY a ring, in general, is meant a ſolid returning into itſelf, of which every ſection perpendicular to the axe, or line paſſing through the middle of the ſolid, is every-where the ſame figure and of the ſame magnitude.

PROBLEM I.

To find the ſurface of a ſolid ring.

RULE.

Multiply the axe by the perimeter of a ſection perpendicular to it, and the product will be the ſurface.

EXAMPLE.

A workman having made for a jeweller a circular ring, or a ring whoſe axe forms the circumference of a circle; it is required to find the expence of the gilding at a penny the ſquare inch, the thickneſs of the ring, or the diameter of a ſection of it, being 4 inches, and the inner diameter, a-croſs from ſide to ſide 18 inches.

Here $18 + 2 = 20 =$ the diameter of the circle formed by the axe; conſequently $20 \times 3{\cdot}14159$ &c. = the length of the axe. But $4 \times 3{\cdot}14159$ &c. = the circumference of a ſection of it. Therefore $20 \times 3{\cdot}14159 \times 4 \times 3{\cdot}14159 = 80 \times 3{\cdot}14159^2 = 789{\cdot}57$ ſquare inches, nearly, $= 789{\cdot}57$ pence $= 3\,l.\ 5\,s.\ 9\frac{1}{2}\,d.$ nearly = the expence required.

PRO-

PROBLEM II.

To find the ſolidity of a ring.

RULE.

Multiply the axe by a ſection perpendicular to it, and the product will be the ſolidity.

EXAMPLE.

Required the price of a ring of iron, whoſe dimenſions are the ſame with that in the example to the laſt problem, at 4 pence a pound; a cubic inch of iron weighing 4·423 ounces averdupois.

Here the area of a ſection being $4^2 \times \cdot 785398$ &c. $= 4 \times 3 \cdot 14159$ &c. which is the ſame number as that expreſſing the circumference, and the axe being the ſame as before, it is evident that the ſolidity will be expreſſed by the ſame number as the ſurface in the laſt example, viz. the ſolidity = 789·57 cubic inches; which multiplied by 4·423 give 3492·2605 ounces = 218·26628 pounds; which, at 4 pence each, come to 3*l.* 12*s.* 9*d.* = the price required.

SCHOLIUM.

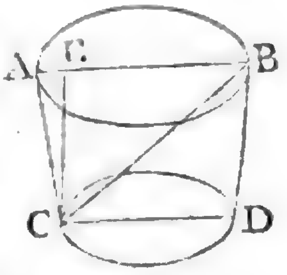
I omit any more examples as the manner of operation is the ſame in all forms with thoſe for priſms, both with regard to the ſurfaces and ſolidities; for, ſince it is evident that any ring is equal to a priſm whoſe altitude and end are reſpectively equal to the axe and ſection of the ring, both in ſurface and ſolidity, the rules for them both muſt be the ſame; and for this reaſon alſo any demonſtration of the rules for rings was quite unneceſſary in this place.

SECTION VIII.

Promiſcuous Queſtions concerning ſolids.

QUESTION I.

REquired the content of a tub whoſe greater diameter is 60, diagonal 66, and the length of the ſtave 30 inches.



$AB = 60 : BC + CA = 96 :: BC - CA = 36 : \frac{36 \times 96}{60} = 6 \times 9{\cdot}6 = 57{\cdot}6 = BE - EA = DC.$

$AE = \frac{AB - CD}{2} = \frac{60 - 57{\cdot}6}{2} = \frac{2{\cdot}4}{2} = 1{\cdot}2.$

$EC = \sqrt{CA^2 - AE^2} = \sqrt{30^2 - 1{\cdot}2^2} = 6\sqrt{5^2 - {\cdot}2^2} = 6\sqrt{25 - {\cdot}04} = 6\sqrt{24{\cdot}96}.$

Then $\overline{AB^2 + AB \times DC + DC^2} \times \frac{1}{3}CE \times {\cdot}785398$ &c. $= \overline{60^2 + 60 \times 57{\cdot}6 + 57{\cdot}6^2} \times 2\sqrt{24{\cdot}96} \times {\cdot}785398$ &c. $= 10373{\cdot}76 \times 4\sqrt{{\cdot}39} \times 3{\cdot}14159$ &c. $= 81410{\cdot}112$ cubic inches $= 288{\cdot}688$ ale gallons = the content required.

QUESTION II.

Three perſons having bought a ſugar loaf, would divide it equally among them by ſections parallel to the baſe; it is required to find the altitude of each perſon's ſhare, ſuppoſing the loaf to be a cone whoſe height is 20 inches.

Similar cones being as the cubes of their altitudes, we ſhall have as $\sqrt[3]{3} : \sqrt[3]{1} :: 20 : 20\sqrt[3]{\frac{1}{3}} = \frac{20\sqrt[3]{9}}{3} = 13{\cdot}8672247$ = the height of the upper part, and as $\sqrt[3]{3} : \sqrt[3]{2} :: 20 : 20\sqrt[3]{\frac{2}{3}} = \frac{20\sqrt[3]{18}}{3} = 17{\cdot}4716107 =$ that of the upper and middle part together; conſequently $17{\cdot}4716107 - 13{\cdot}8672247 = 3{\cdot}604386 =$ the height of the middle part, and $20 - 17{\cdot}4716107 = 2{\cdot}5283893 =$ that of the lower part.

QUESTION III.

A ſilver cup, in the form of the fruſtum of a cone, whoſe top diameter is 3 inches, its bottom diameter 4, and its altitude 6 inches, being filled with liquor, a perſon drank out of it until he could ſee the middle of the bottom; it is required to find how much he drank.

By problem 14 of ſection 1 we have $D = 4$, $d = 3$, $b = 6$, and $BD = \frac{1}{2}D = 2$; hence $D^3 - d^3 = 64 - 27 = 37$, $P =$ (the tabular area whoſe verſed ſine is $\frac{BD}{D}$ or $\frac{1}{2} =$) $\frac{1}{2}n = \frac{1}{2}$ of $\cdot 78539816$ &c. $Q =$ (the tabular area whoſe verſed ſine is $\frac{BD - D + d}{d}$ or $\frac{1}{3} =$) $\cdot 22945528$, $\frac{BD}{BD - D + d} = 2$, and $\frac{\frac{1}{3}b}{D - d} = 2$; conſequently $\overline{37n - 32n + 54Q\sqrt{2}} \times 2 = \overline{5n + 54Q\sqrt{2}} \times 2 = 7{\cdot}8539816 + 35{\cdot}0458624 = 42{\cdot}899844$ cubic inches $= \cdot 152127$ ale gallons $=$ the quantity required, or 1 gill and $\frac{1}{5}$ nearly.

QUESTION IV.

I have a right cone which coſt me 5*l.* 13*s.* 7*d.* at 10*s.* a cubic foot, the diameter of its baſe being to its altitude as 5 to 8; and would have its

convex

convex ſurface divided in the ſame ratio by a plane parallel to the baſe, the upper part to be the greater: Required the ſlant height of each part.

Here $\frac{5l.\ 13s.\ 7d.}{10s.} = \frac{1363}{120}$ = the ſolidity of the cone in feet, and $\cdot 785398\ \&c. \times 5^2 \times \frac{8}{3} = 200 \times \frac{\cdot 785398\ \&c.}{3} = \frac{200n}{3}$ = that of a cone ſimilar to it whoſe altitude is 8.

Now, the ſurfaces of ſimilar ſolids being as the ſquares of their like dimenſions, we have $\sqrt{5+8} : \sqrt{8} :: \sqrt{2 \cdot 5^2 + 8^2}$ = the ſide of the ſaid ſimilar cone $: \sqrt{\frac{562}{13}}$ = the ſlant height of the upper part of this cone when its ſurface is divided in the propoſed ratio; and conſequently $\sqrt{70\frac{1}{4}} - \sqrt{\frac{562}{13}} = \sqrt{\frac{562}{8}} - \sqrt{\frac{562}{13}}$ = that of the under part of it.

Then, becauſe ſimilar ſolids are as the cubes of their like dimenſions, we ſhall have $\sqrt[3]{\frac{200n}{3}} : \sqrt[3]{\frac{1363}{120}} :: \sqrt{\frac{562}{13}} : \frac{1}{20}\sqrt[3]{\frac{1363}{n}} \times \sqrt{\frac{562}{13}} = 3{\cdot}9506486$ = the ſlant height of the upper part required, and $\sqrt[3]{\frac{200n}{3}} : \sqrt[3]{\frac{1363}{120}} :: \sqrt{\frac{562}{8}} - \sqrt{\frac{562}{13}} : \frac{1}{20}\sqrt[3]{\frac{1363}{n}} \times \overline{\sqrt{\frac{562}{8}} - \sqrt{\frac{562}{13}}} = \frac{1}{20}\sqrt[3]{\frac{1363}{n}} \times \sqrt{\frac{562}{8}} - 3{\cdot}9506486 = 5{\cdot}0361098 - 3{\cdot}9506486 = 1{\cdot}0854612$ = that of the under part.

QUEST.

QUESTION V.

There is a mill-hopper in the form of a ſquare pyramid, whoſe ſolid content is $13\frac{1}{2}$ feet; but one foot is cut off its perpendicular altitude, to make a paſſage for the grain from the fruſtum or hopper to the mill-ſtone: The ſides of its greater and leſs end are in proportion as $4\frac{1}{2}$ to 1. Required the content of the fruſtum, in corn meaſure, in which 268·8025 cubic inches make a gallon, or 2150·42 = the buſhel.

Since ſimilar ſolids are as the cubes of their like dimenſions, we ſhall have $4{\cdot}5^3 : 1^3 :: 13\frac{1}{2} : \frac{13\frac{1}{2}}{4{\cdot}5^3} = \frac{3}{4{\cdot}5^2} = \frac{12}{81} = \frac{4}{27}$ = the content of the part cut off.

Wherefore $13\frac{1}{2} - \frac{4}{27} = \frac{27}{2} - \frac{4}{27} = \frac{27^2 - 8}{54} = \frac{729 - 8}{54} = \frac{721}{54}$ cubic feet $= \frac{721}{54} \times 1728 = \frac{721 \times 288}{9} = 721 \times 32 = 23072$ cubic inches = the content of the hopper. Which being divided by 2150·42 gives 10·729067 buſhels for the content in corn meaſure.

QUESTION VI.

A piece of round tapering timber, whoſe top and bottom diameters are 40 and 50 inches, and its height 6 feet, is to be cut through the extremity of the leſs diameter and parallel to the tapering direction; required the content of the two parts or hoofs into which it is cut.

By prob. 14 ſect. 1, $D = 50$, $d = 40$, $h = 6$ feet = 72 inches; then $A = 50^2 \times$ tabular ſegment whoſe height is $\left(\frac{50-40}{50}=\frac{10}{50}=\right)\cdot2=50^2\times\cdot1118238$ $=279\cdot5595$.

Hence $\overline{\frac{A\times D}{D-d}-\frac{4}{3}d\sqrt{\overline{D-d}\times d}}\times\frac{1}{3}b=$ $\overline{\frac{279\cdot5595\times50}{50-40}-\frac{160}{3}\sqrt{\overline{50-40}\times40}}\times24=$ $\overline{279\cdot5595\times5-\frac{160-20}{3}}\times24=\overline{279\cdot5595\times3-160\times4}$ $\times40=198\cdot6785\times40=7947\cdot14=$ the content of the hoof cut off.

But, by prob. 8 ſect. 1, $\frac{50^3-40^3}{50-40}\times\frac{72}{3}\times\cdot785398$ &c. $=\overline{5^3-4^3}\times100\times24\times\cdot785398$ &c. $=166400$ $\times\cdot785398$ &c. $=250690\cdot2543893=$ the content of the whole piece.

Conſequently $130690\cdot2543893-7947\cdot14=$ $122743\cdot1143893=$ the ſolidity of the complemental hoof.

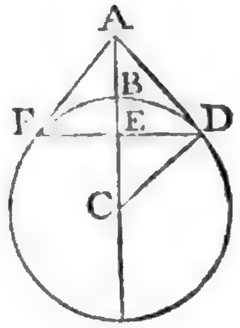
QUESTION VII.

"Two Oxonians, meeting at an inn, encountered with a tankard of negus; the one, being pot-valiant, gave it a black-eye, as it is called, that is, he drank till he could ſee the center of the bottom of the tankard; the other drank the reſt: Now, if the liquor coſt 1*s.* 6*d.* and the tankard meaſure 4 inches diameter at the top and bottom, and 6 inches in depth, what muſt each perſon pay proportionable to the liquor he drank."

By prob. 4 ſect. 1, we have $\frac{1}{3}ddh$ for the quantity left by the firſt perſon; and, by prob. 2 ſect. 1, $nddh=$ the content of the whole tankard, d being the diameter, b the height, and $n = {}^{\cdot}785398$ &c. that is, the whole is to the leſs ſhare as n to $\frac{1}{6}$; conſequently as $n : \frac{1}{6} :: 18d. : \frac{3}{n} = \frac{3}{{}^{\cdot}785398 \text{ \&c.}} =$ $3{\cdot}8197185$ pence = the ſum the latter perſon muſt pay; and hence $18d. - 3{\cdot}8197185d. = 14{\cdot}18028 15$ pence = the ſum the firſt drinker muſt pay.

QUESTION VIII.

How many acres of the earth's ſurface may be ſeen from the top of a ſteeple whoſe height is 400 feet; the earth being ſuppoſed a perfect ſphere whoſe circumference is 25000 miles.



Having drawn from A, the top of the ſteeple, two lines AD, AF, to touch the earth whoſe center is C, and the other lines as in the figure; FBD will be the ſurface required.

By ſimilar triangles, $AC : CD$ or $CB :: CD$ or $CB : CE$, hence $AC : AC - CB = BA :: CB : CB - CE =$

$$EB = \frac{CB \times BA}{AC} = \frac{CB \times BA}{CB + BA} = \frac{\frac{25000}{3{\cdot}14159 \text{ \&c.}} \times \frac{400}{5280}}{\frac{25000}{3{\cdot}14159 \text{ \&c.}} + \frac{800}{5280}} =$$

$$\frac{25000 \times 4}{2500 \times 528 + 3{\cdot}14159 \text{ \&c.} \times 8} = \frac{100000}{1320025{\cdot}1327412287} =$$

${}^{\cdot}0757561333641 6945$.

Then, by problem 19 ſection 1, we have ${}^{\cdot}07575613336416945 \times 250000 \times 640$ (the acres in a ſquare mile) $= 12120981{\cdot}338267112$ = the number of acres required.

QUESTION IX.

How high above the ſurface of the earth muſt a perſon be raiſed, that he may ſee one-third of its ſurface.

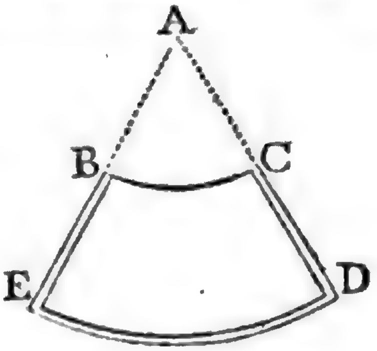
The ſurfaces of ſegments being, by prob. 19 ſect. 1, as their altitudes, the altitude of the ſegment in the queſtion muſt be one-third of the diameter or two-thirds of the radius of the ſphere; that is, $EB = \frac{2}{3} BC$; and conſequently $EC = \frac{1}{3} CB$. [See the laſt figure]

But, by ſimilar triangles, as $EC : CD$ or $CB :: CD$ or $CB : CA$, and hence $EC : CB - CE = EB :: BC : CA - CB = BA = \frac{CB \times BE}{EC}$, which is a general expreſſion for the height above the ſurface, and which when, as above, BE is $= 2EC$, becomes $2BC$ = the diameter of the earth; which is the height required.

QUESTION X.

If from a piece of tin AED, in the form of the ſector of a circle, whoſe radius AE or AD is 30 inches, and the length of its arc ED 36 inches, be cut another ſector ABC whoſe radius AB or AC is 20 inches; and if then the remaining fruſtum $BCDE$ be rolled up ſo as to form the fruſtum of a cone, it is required to find its content, ſuppoſing one-eighth of an inch to be allowed of its ſlant height BE for the bottom, and the ſame allowance off the circumference of both top and bottom for what the ſides BE, CD, fold over each other, in order to their being ſoldered together.

By



By ſimilar figures, as $EA = 30 : AB = 20 :: ED = 36 : \frac{36 \times 20}{30} = 24 = BC$. Then the circumferences of the top and bottom are $(36 - \frac{1}{8} = 35\frac{7}{8} =)$ $35{\cdot}875$, and $23{\cdot}875$; and the ſide of the veſſel is $(30 - 20 - \frac{1}{8} = 9\frac{7}{8} =)$ $9{\cdot}875$.

Hence the diameters of the ends will be $D = \frac{35{\cdot}875}{3{\cdot}14159}$, and $d = \frac{23{\cdot}875}{3{\cdot}14159}$; half their difference is $\frac{6}{3{\cdot}14159}$; conſequently the perpendicular altitude will be found, by the property of right-angled triangles, to be $\sqrt{9{\cdot}875^2 - \frac{6^2}{3{\cdot}14159^2}} = \frac{\sqrt{9{\cdot}875^2 \times 3{\cdot}14159^2 - 6^2}}{3{\cdot}14159 \text{ \&c.}}$ $= \frac{30{\cdot}43749}{3{\cdot}14159 \text{ \&c.}}$.

Then, by problem 8 ſection 1, we ſhall have

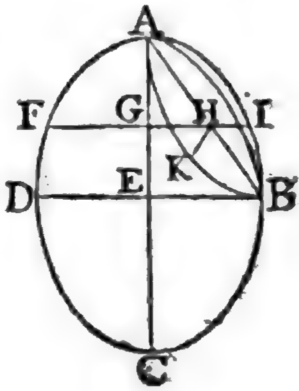
$$\frac{35{\cdot}875^2 + 35{\cdot}875 \times 23{\cdot}875 + 23{\cdot}875^2}{3{\cdot}14159^2} \times \frac{30{\cdot}43749}{3{\cdot}14159 \text{ \&c.}} \times \frac{3{\cdot}14159 \text{ \&c.}}{12}$$

$$= \frac{287^2 + 287 \times 191 + 191^2}{8^2 \times 3{\cdot}14159^2} \times \frac{30{\cdot}43749}{12} = \frac{170667}{768} \times \frac{30{\cdot}43749}{3{\cdot}14159^2}$$

$= 685{\cdot}3263$ cubic inches $=$ the content required.

QUESTION XI.

It is required to find the area of the ſection of any ſpheroid, formed by a plane paſſing through the extremities of the two axes, the one axe being 80 and the other 60.



By prop. 1 ſect. 2 the ſection AKB will be an ellipſe whoſe axes are AB and $2HK$, H being the middle of AB.

But $AB = \sqrt{BE^2 + EA^2} = \sqrt{30^2 + 40^2} = 50$.

And, having drawn $FGHI$ parallel to DB, by the nature of the ellipſe, $AE^2 : EB^2 :: AE^2 - EG^2 = AE^2 - \frac{1}{4}AE^2 = \frac{3}{4}AE^2 : GI^2 = \frac{3}{4}EB^2$; and hence, by the nature of the circle, $2\sqrt{FH \times HI} = 2\sqrt{IG^2 - GH^2} = 2\sqrt{\frac{3}{4}EB^2 - \frac{1}{4}EB^2} = EB\sqrt{2} =$ the other axe $2HK$.

Conſequently $AB \times 2HK \times {\cdot}785398$ &c. $= 50 \times 30\sqrt{2} \times {\cdot}785398$ &c. $= 1666{\cdot}081101807 =$ the area of the ſection when the ſpheroid is oblong. And $50 \times 40\sqrt{2} \times {\cdot}785398$ &c. $= 2221{\cdot}441469076 =$ the area when oblate.

QUESTION XII.

There is a punch bowl in the form of the ſegment of an oblong ſpheroid whoſe axes are to each other in the proportion of 3 to 4, the depth of the bowl being one-fourth of the whole tranſverſe axe, and the diameter of its top 20 inches: it is required to determine what number of rounds a company of 10 men may take out of it, when filled with liquor, uſing a conical glaſs whoſe depth is 2 inches, and the diameter of its top one inch and a half.

The ſegment, whoſe altitude is 1, of the ſpheroid whoſe axes are 3 and 4 is ſimilar to the propoſed one; and, by the nature of the ellipſe, $4 : 3 :: 2\sqrt{3 \times 1} : \frac{3\sqrt{3}}{2} =$ the diameter of its top; but by prob. 14 ſect. 3, its content is $\frac{6-1}{3 \times 4^2} \times 3^2 \times 3{\cdot}14159$ &c. $= \frac{15 \times {\cdot}785398 \text{ \&c.}}{4}$; and ſimilar ſolids are as the cubes of their like dimenſions; wherefore $\overline{\frac{3\sqrt{3}}{2}}\Big|^3 : 20^3 :: \frac{15 \times {\cdot}785398 \text{ \&c.}}{4} : \frac{40^3}{3^4 \times \sqrt{3}} \times \frac{15 \times {\cdot}785398 \text{ \&c.}}{4} = \frac{80000\sqrt{3}}{81} \times {\cdot}785398 \text{ \&c.}$ = the content of the bowl.

But $1{\cdot}5^2 \times 2 \times \frac{{\cdot}785398 \text{ \&c.}}{3} = \frac{3}{2} \times {\cdot}785398 \text{ \&c.}$ = the content of the glaſs, and $15 \times {\cdot}785398$ &c. = each round.

Wherefore $\frac{80000\sqrt{3} \times {\cdot}785398 \text{ \&c.}}{81 \times 15 \times {\cdot}785398 \text{ \&c.}} = \frac{16000\sqrt{3}}{243} = 114{\cdot}0444976$ = the number of rounds required.

QUESTION XIII.

Two perſons would divide between them, by a plane perpendicular to the baſe, an hay-rick in the form of a paraboloid whoſe altitude is 40 and the diameter of its baſe 30 feet; it is required to find the difference between the ſolidities of the parts, ſuppoſing the altitude of the ſection to be 28 feet.

Here $DH = 40$ [See fig. to prob. 13 ſect. 4], $EA = 30$, and $BC = HL = 28$.——Hence $LD = DH - HL = 40 - 28 = 12$; and, by prob. 3 ſect. 4, as $\sqrt{HD} : \sqrt{DL} :: HA : LC = HB = \frac{15\sqrt{12}}{\sqrt{40}} = \frac{3\sqrt{30}}{2}$; hence $\sqrt{AH^2 - HB^2} = \sqrt{15^2 - \frac{9 \times 30}{4}} = \frac{315}{2}$, and $\frac{BA}{AE} = \frac{15 - \frac{3\sqrt{30}}{2}}{30} = \frac{10 - \sqrt{30}}{20} = {\cdot}2261387$ = the tabular verſed ſine ſimilar to AB, whoſe tabular area is ·13322474, which taken from ·39269908, the tabular ſemi-circle, leaves ·25947434 = the tabular area ſimilar to double the area $HBGK$; conſequently

ſequently $\frac{1}{2} \times 30^2 \times {}^{\cdot}25947434 = 116{\cdot}763453 =$ the area *HBGK*.

Then, by cor. 4 to prob. 13 ſect. 1, we ſhall have $116{\cdot}763453 \times 40 \times 2 + \frac{315\sqrt{28 \times 12}}{2} = 9341{\cdot}07624 + 630\sqrt{21} = 9341{\cdot}07624 + 2887{\cdot}022691 = 12228{\cdot}09893$ = the difference required.

QUESTION XIV.

The curve ſurface of a paraboloid being $\overline{109\sqrt{109}-27} \times 2 \times 3{\cdot}14159$ &c. $= 6980{\cdot}57746$ &c. it is required to find its altitude and the diameter of it baſe, ſuppoſing them in proportion as 5 to 6.

By prob. 7 ſect. 4, $\frac{\overline{3^2+10^2}|^{\frac{3}{2}} - 3^3}{3^2+10^2-3^2} \times 3 \times \frac{2}{3} \times 3{\cdot}14159$ &c. $= \frac{109\sqrt{109}-27}{100} \times 2 \times 3{\cdot}14159$ &c. is the ſurface of a paraboloid whoſe altitude is 5 and baſe diameter 6, which is ſimilar to the propoſed paraboloid; but the ſurfaces of ſimilar bodies are as the ſquares of their lineal dimenſions, or the demenſions are as the roots of the ſurfaces; wherefore as $\sqrt{\frac{109\sqrt{109}-27}{100}} \times 2 \times 3{\cdot}14159$ &c. is to $\sqrt{\frac{109\sqrt{109}-27}{1}} \times 2 \times 3{\cdot}14159$ &c. or as 1 : 10 :: 5 : 50 = the altitude, and :: 6 : 60 = the baſe-diameter required.

QUESTION XV.

If a cubical foot of braſs were to be drawn into wire, of one-fortieth of an inch in diameter, it is required to determine the length of the ſaid wire, allowing no loſs in the metal.

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Here $1 \times 12^3 = 1728$ = the ſolidity of the wire in inches; and $\overline{\frac{1}{40}}\Big|^2 \times {}^{\cdot}785398$ &c. $= \frac{1}{1600} \times {}^{\cdot}785398$ &c. the area of its end; conſequently $\frac{1728}{\frac{1}{1600} \times {}^{\cdot}785398 \text{ \&c.}}$ $= \frac{1728 \times 1600}{{}^{\cdot}785398 \text{ \&c.}} = 3520252{}^{\cdot}69329$ inches = the length required.

QUESTION XVI.

A gentleman having a bowling green, 300 feet long and 200 feet broad, which he would raiſe one foot higher by means of the earth to be dug out of a ditch around it; it is required to find to what depth the ditch muſt be dug, its breadth being every where 8 feet.

Here the ditch is a kind of ring whoſe breadth is 8, and length $= 300 \times 2 + 200 \times 2 + 8 \times 4 = 516 \times 2 = 1032$, and conſequently the area of its bottom $= 8 \times 1032 = 8256$; but $300 \times 200 \times 1 = 60000$ is its content; conſequently $\frac{60000}{8256} = 7\frac{23}{86}$ feet = the depth required.

QUESTION XVII.

Required the weight of a bomb-ſhell, or hollow ſphere of caſt iron, whoſe out-ſide diameter is 15, and the thickneſs of the metal $2\frac{1}{2}$ inches; the gravity of water being to that of caſt iron as 1 to 7, and a cubic inch of water weighing ·5787 ounces avoirdupois.

Here $15^3 \times \frac{3{}^{\cdot}14159 \text{ \&c.}}{6}$ = the ſolidity of the whole ſphere, ſuppoſing it all ſolid; and, ſince $15 - 2\frac{1}{2} \times 2 = 15 - 5 = 10$ is the diameter of the cavity, $10^3 \times \frac{3{\cdot}14159 \text{ \&c.}}{6}$ will be the content of the cavity; conſequently their difference $\overline{15^3 - 10^3} \times \frac{3{\cdot}14159 \text{ \&c.}}{6} = \overline{3^3 - 2^3} \times 5^3 \times \frac{3{\cdot}14159 \text{ \&c.}}{6} = 19 \times 125 \times \frac{3{\cdot}14159 \text{ \&c.}}{6}$ = the ſolidity of the metal in inches.

But $1 : 7 :: {\cdot}5785 : 4{\cdot}0509$ ounces = the weight of a cubic inch of the metal.

Conſequently $4{\cdot}0509 \times 19 \times 125 \times \frac{3{\cdot}14159 \text{ \&c.}}{6} = 1{\cdot}3503 \times 19000 \times \frac{{\cdot}785398 \text{ \&c.}}{4} = 6413{\cdot}925 \times {\cdot}785398$ &c. $= 5037{\cdot}4849152$ ounces $= 314{\cdot}8428072$ pounds avoirdupois = the weight required.

QUESTION XVIII.

Of what diameter muſt the bore of a cannon be caſt for a ball of 24 pound weight, ſo that the diameter of the bore may be one tenth of an inch more than that of the ball.

By the laſt, the weight of a cubic inch of caſt iron is $4{\cdot}0509$ ounces, conſequently as $4{\cdot}0509$ ounces : 1 inch :: 24 lb. or 24×16 oz. : $\frac{24 \times 16}{4{\cdot}0509} = \frac{128}{1{\cdot}3503}$ inches = the ſolidity of the ball; but the ſolidity is equal to the cube of the diameter multiplied by $\frac{3{\cdot}14159 \text{ \&c.}}{6}$; wherefore $\sqrt[3]{\frac{128 \times 6^3}{1{\cdot}3503 \times 3{\cdot}14159 \text{ \&c.}}} = \sqrt[3]{\frac{256}{{\cdot}4501 \times 3{\cdot}14159 \text{ \&c.}}} = 8 \times \sqrt[3]{\frac{1}{{\cdot}9002 \times 3{\cdot}14159 \text{ \&c.}}} = 5{\cdot}657098$ inches = the diameter of the ball; to which adding ${\cdot}1$ makes $5{\cdot}757098$ inches = the diameter required.

PART

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PART IV.

Having given at large the menſuration of all figures which are generally thought to be met with in real practice, viz. the figures formed by right or circular lines, or by the conic ſections; I ſhall, under this part, give a *very brief* treatiſe of ſuch other things as belong to menſuration in general; ſuch as the quadrature and cubature of figures from general equations expreſſing their nature or property, the method of equidiſtant ordinates, and the relation between the areas or ſolidities of figures and their centers of gravity.

SECTION I.

Of the true QUADRATURE *and* CUBATURE *of* CURVES *in* GENERAL.

PROPOSITION I.

If z *be the abſciſſa of any curve, and the ordinate* y *be equal to* $z^{p-1}\times\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\delta z^{3n}+\varepsilon z^{4n}+\&c.}\big|^{q-1}$ $\times : a+bz^{n}+cz^{2n}+dz^{3n}+ez^{4n}$ *&c. and if* $\frac{p}{n}$ *be put equal to* r, $r+q=s$, $s+q=t$, $t+q=v$, $v+q=w$, *&c. I ſay the area will be equal to* $z^{p}\times\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\delta z^{3n}+\varepsilon z^{4n}+\&c.}\big|^{q}$ *drawn into the ſeries*

$$+\frac{\frac{a}{n}}{\alpha r}$$

$$+\frac{\frac{b}{n}-A\beta s}{\alpha.\overline{r+1}}z^{n}$$

$$+\frac{\frac{c}{n}-A\gamma t-B\beta.\overline{s+1}}{\alpha.\overline{r+2}}z^{2n}$$

$$+\frac{\frac{d}{n}-A\delta v-B\gamma.\overline{t+1}-C\beta.\overline{s+2}}{\alpha.\overline{r+3}}z^{3n}$$

$$+\frac{\frac{e}{n}-A\epsilon w-B\delta.\overline{v+1}-C\gamma.\overline{t+2}-D\beta.\overline{s+3}}{\alpha.\overline{r+4}}z^{4n}$$

Where A, B, C, *&c. are the whole coefficients of the preceding terms with their ſigns* + *or* −; *viz.* $A=\frac{\frac{a}{n}}{\alpha r}$, $B=\frac{\frac{b}{n}-A\beta s}{\alpha.\overline{r+1}}$, *&c.*

DEMONSTRATION.

The fluxion of the area is $y\dot{z}=z^{p-1}\dot{z}\times\overline{\alpha+\beta z^{n}+\gamma z^{2n}\ \&c.}|^{q-1}\times:a+bz^{n}+cz^{2n}+\&c.$

Let the fluent of this expreſſion, or the area required, be repreſented by $z^{p}\times\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\&c.}|^{q}\times:A+Bz^{n}+Cz^{2n}+\&c.$ where *A*, *B*, *C*, &c. are not ſuppoſed to repreſent the ſame quantities as in the propoſition, but other quantities yet to be determined.

Then let the fluxion of this laſt aſſumed area or fluent be taken, and compared with the given fluxion, ſo ſhall we have equations for determining the values of the aſſumed letters *A*, *B*, *C*, &c.

Thus the fluxion of the aſſumed area being

$$z^{p-1}\dot{z}\times\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\&c.}\Big|^{q}\times:Ap+\overline{p+n}.Bz^{n}+\overline{p+2n}.Cz^{2n}+\&c.+nqz^{p-1}\dot{z}\times\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\&c.}\Big|^{q-1}\times:\beta z^{n}+2\gamma z^{2n}+3\delta z^{3n}+\&c.\times:A+Bz^{n}+Cz^{2n}+\&c.$$

Or, by proper multiplication, &c.

$$z^{p-1}\dot{z}\times\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\&c.}\Big|^{q-1}\text{ drawn into}$$

$$\begin{aligned}&p\alpha A+\overline{p+qn}.\beta Az^{n}+\overline{p+2qn}.\gamma Az^{2n}+\overline{p+3qn}.\delta Az^{3n}+\&c.\\&\quad+\overline{p+n}.\alpha Bz^{n}+\overline{p+qn+n}.\beta Bz^{2n}+\overline{p+2qn+n}.\gamma Bz^{3n}\\&\qquad+\overline{p+2n}.\alpha Cz^{2n}+\overline{p+qn+2n}.\beta Cz^{3n}\\&\qquad\quad+\overline{p+3n}.\alpha Dz^{3n}\end{aligned}$$

If the ſeveral terms of this ſeries be compared with the correſponding terms of the ſeries in the given fluxion, we ſhall obtain theſe equations, viz.

$a=p\alpha A$,

$b=\overline{p+qn}.\beta A+\overline{p+n}.\alpha B$,

$c=\overline{p+2qn}.\gamma A+\overline{p+qn+n}.\beta B+\overline{p+2n}.\alpha C$,

$d=\overline{p+3qn}.\delta A+\overline{p+2qn+n}.\gamma B+\overline{p+qn+2n}.\beta C+\overline{p+3n}.\alpha D$,

$e=\overline{p+4qn}.\epsilon A+\overline{p+3qn+n}.\delta B+\overline{p+2qn+2n}.\gamma C+\overline{p+qn+3n}.\beta D+\overline{p+4n}.\alpha E$,

&c. &c.

From whence we obtain

$$A = \frac{a}{\alpha p},$$

$$B = \frac{b - \overline{p + qn}.\beta A}{\alpha . \overline{p+n}},$$

$$C = \frac{c - \overline{p + 2qn}.\gamma A - \overline{p + qn + n}.\beta B}{\alpha . \overline{p + 2n}},$$

$$D = \frac{d - \overline{p + 3qn}.\delta A - \overline{p + 2qn + n}.\gamma B - \overline{p + qn + 2n}.\beta C}{\alpha . \overline{p + 3n}},$$

$$E = \frac{e - \overline{p + 4qn}.\epsilon A - \overline{p + 3qn + n}.\delta B - \overline{p + 2qn + 2n}.\gamma C - \overline{p + qn + 3n}.\beta D}{\alpha . \overline{p + 4n}},$$

&c.

Where now the letters *A*, *B*, *C*, &c. in the terms on the right ſide of theſe equations denote the preceding terms, as ſpecified in the propoſition.

And if $\frac{p}{n}$ be put $= r$, $r + q = s$, $s + q = t$, $t + q = v$, &c. theſe laſt equations will become

$$A = \frac{\frac{a}{n}}{\alpha r},$$

$$B = \frac{\frac{b}{n} - A\beta s}{\alpha . \overline{r + 1}},$$

$$C = \frac{\frac{c}{n} - A\gamma t - B\beta . \overline{s + 1}}{\alpha . \overline{r + 2}},$$

$$D = \frac{\frac{d}{n} - A\delta v - B\gamma . \overline{t + 1} - C\beta . \overline{s + 2}}{\alpha . \overline{r + 3}},$$

$$E = \frac{\frac{e}{n} - A\epsilon w - B\delta . \overline{v + 1} - C\gamma . \overline{t + 2} - Db . \overline{s + 3}}{\alpha . \overline{r + 4}},$$

&c.

Which values of A, B, C, &c. being ſubſtituted for them in the aſſumed fluent or area, will give the area as in the propoſition. *Q.E.D.*

And much after the ſame manner we may find the area of the curve whoſe abſciſſa is z and ordinate

$$\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\&c.}\times z^{p-1}\times\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\delta z^{3n}+\&c.}\Big|^{q-1}$$

$$\times\overline{A+Bz^{n}+Cz^{2n}+Dz^{3n}+\&c.}\Big|^{r-1}\times\&c.$$ whatever be the number of ſeries.

When, after ſome of the firſt terms, the numerators of each of the following terms of the ſeries $\dfrac{\frac{a}{n}}{\alpha r}+\dfrac{\frac{b}{n}-A\beta s}{\alpha.\overline{r+1}}z^{n}+\&c.$ become equal to nothing, the ſeries will break off and terminate, and then the curve is ſaid to be quadrable; if otherwiſe, it is ſaid to be non-quadrable.——If r be either nothing or a negative integer number, it is evident that the denominator of one of the terms of the above ſeries will become equal to nothing, and then that term will be infinite; and if this happen before the ſeries terminate by means of the numerators becoming equal to nothing, the value of the area comes out infinite, in which caſe the ſeries is ſaid to fail.

The curve is denominated from the number of terms contained in the quantity $\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\delta z^{3n}+\&c.}\Big|^{q}$; ſo, if it contain only one term α, it is a ſimple nominal; if it contain two terms $\overline{\alpha+\beta z^{n}}\Big|^{q}$, it is a binomial; if three

$\overline{\alpha+\beta z^{n}+\gamma z^{2n}}|^{q}$, it is a trinomial; &c. But in what follows I ſhall conſider none beyond the trinomial.

The area might be expreſſed by a deſcending ſeries, and from thence other caſes of the termination of the ſeries might be pointed out; but this I ſhall do in the particular forms, as it will in them be done with greater eaſe.

Nor ſhould it be wondered at that the area admits of two different values; for when an ordinate flows, the area on one ſide of it will increaſe as faſt as that on the other decreaſes, conſequently the fluxions of thoſe two areas will be equal to each other, that is, the fluxions of the areas are both expreſſed by the ſame quantity, and it is therefore but right that the fluxion ſhould admit of two different fluents anſwering to the two areas on each ſide of the ordinate. When the expreſſion comes out affirmative, it denotes the area lying on that ſide of the ordinate from which it is ſuppoſed to move; but when it comes out negative, it denotes the area on the other ſide of the ordinate. When it comes out infinite, it denotes the area lying along the abſciſſa infinitely produced.

If all the terms of the ſeries $a+bz^{n}+cz^{3n}+\&c.$ after the firſt vaniſh, and by that means the expreſſion for the ordinate become only $az^{p-1}\times\overline{\alpha+\beta z^{n}+\gamma z^{2n}+\&c.}|^{q-1}$, the curve or area is ſaid to be ſimple; but if there be more terms than one, it is ſaid to be compound.

In what follows I ſhall conſider chiefly thoſe caſes in which the curve is ſaid to be ſimple; not only becauſe thoſe are the caſes that generally happen in practice, but becauſe every compound caſe may be reſolved into ſo many ſimple ones as there are terms in the ſeries $a+bz^{n}+cz^{2n}+\&c.$ and then by finding the area for every ſimple caſe, the aggregate of thoſe areas will be the area for the whole compound caſe.

In the quantity expreſſing any area write that particular value of the abſciſſa which it is ſuppoſed to have where the area commences or when it is equal to nothing, and the value of the area reſulting from that ſubſtitution will be equal to nothing if the firſt area were rightly aſſigned, and then it needs no correction; but if the area by this ſubſtitution come out of ſome value, by juſt ſo much will the firſt area differ from the truth, and muſt be corrected by ſubtracting the ſaid value from it.

It may alſo be obſerved that when the ordinate is oblique to the abſciſſa, the area as found by the ſeries muſt be drawn into the ſine of the angle of inclination of the ordinate to the abſciſſa.

EXAMPLE.

If an ordinate y be $= \dfrac{3a-bzz}{zz\sqrt{az-bz^{3}+cz^{4}}}$.

To have this in the ſame form with the general ſeries, we muſt expreſs it thus $y=z^{-\frac{3}{2}-1}\times\overline{3a-bzz}\times\overline{a-bz^{2}+cz^{3}}|^{\frac{1}{2}-1}$; or it may be expreſſed thus $y=z^{-1-1}\times\overline{-b+3az^{-2}}\times\overline{c-bz^{-1}+az^{-3}}|^{\frac{1}{2}-1}$.

Now by comparing the firſt of theſe forms with the general expreſſion of the area, we obtain $a = 3a$, $b = 0$, $c = -b$; $\alpha = a$, $\beta = 0$, $\gamma = -b$, $\delta = c$; $p = -\frac{3}{2}$, $n = 1$, $q = \frac{1}{2}$.

Hence $r = \frac{p}{n} = -\frac{3}{2}$, $s = r + q = -1$, $t = s + q = -\frac{1}{2}$, $v = t + q = 0$, $w = v + q = \frac{1}{2}$, &c.

Then, by ſubſtituting theſe values in the general ſeries, we have $z^{-\frac{1}{2}} \times \overline{a - bz^2 + cz^3}|^{\frac{1}{2}} \times -2$ (all the reſt of the terms after the firſt vaniſhing) $= -2\sqrt{\frac{a - bz^2 + cz^3}{z^3}}$. And becauſe this expreſſion is negative, it denotes the area on the other ſide of the ordinate.

Again, by comparing the latter form with the general ſeries, we obtain $a = -b$, $b = 0$, $c = 3a$; $\alpha = c$, $\beta = -b$, $\gamma = 0$, $\delta = a$; $p = -1$, $n = -1$, $q = \frac{1}{2}$.

Hence $r = \frac{p}{n} = 1$, $s = r + q = \frac{3}{2}$, $t = s + q = 2$, &c.

Then ſubſtituting theſe values in the general ſeries, we obtain $\sqrt{\frac{a - bz^2 + cz^3}{z^5}} \times : \frac{b}{c} + \frac{3bA}{4cz} + \frac{5bB - 6a}{6cz^2} + \frac{7bC - 5aA}{8cz^3} + \frac{9bD - 7aB}{10cz^4} +$ &c. for the area in this caſe; where the law of the progreſſion is manifeſt, and where A, B, C, &c. denote the whole coefficients of the firſt, ſecond, third, &c. terms.

COROLLARY I.

When b, c, d, &c. are each equal to nothing, the curve will be ſimple, and the general expreſſion for ſimple

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ple areas becomes $\frac{az^p}{\alpha n} \times \overline{\alpha+\beta z^n+\gamma z^{2n}+\delta z^{3n}+\&c.}\Big|^{q} \times$

$$
\begin{aligned}
&+\frac{1}{r}\\
&-\frac{s\beta A}{\overline{r+1}.\alpha}z^n\\
&-\frac{t\gamma A+\overline{s+1}.\beta B}{\overline{r+2}.\alpha}z^{2n}\\
&-\frac{v\delta A+\overline{t+1}.\gamma B+\overline{s+2}.\beta C}{\overline{r+3}.\alpha}z^{3n}\\
&-\frac{w\epsilon A+\overline{v+1}.\delta B+\overline{t+2}.\gamma C+\overline{s+3}.\beta D}{\overline{r+4}.\alpha}z^{4n}\\
&\quad\&c.
\end{aligned}
$$

COROLLARY II.

If the curve be only a trinomial, as $y = az^{p-1} \times \overline{\alpha+\beta z^n+\gamma z^{2n}}\Big|^{q-1}$, that is, δ, ϵ, &c. each equal to nothing, the area in the laſt corollary will become $\frac{az^p}{\alpha n} \times \overline{\alpha+\beta z^n+\gamma z^{2n}}\Big|^{q} \times : \frac{1}{r} - \frac{s\beta A}{\overline{r+1}.\alpha}z^n - \frac{\overline{s+1}.\beta B+t\gamma A}{\overline{r+2}.\alpha}z^{2n} - \frac{\overline{s+2}.\beta C+\overline{t+1}.\gamma B}{\overline{r+3}.\alpha}z^{3n} - \frac{\overline{s+3}.\beta D+\overline{t+2}.\gamma C}{\overline{r+4}.\alpha}z^{4n} - \&c.$

But to find another expreſſion of the area in a deſcending ſeries, we have the ordinate y or $az^{p-1} \times \overline{\alpha+\beta z^n+\gamma z^{2n}}\Big|^{q-1} = az^{p-1+2nq-2n} \times \overline{\gamma+\beta z^{-n}+\alpha z^{-2n}}\Big|^{q-1}$; then by comparing this with the original ſeries, the

the area will come out $\frac{az^{p-2n}}{\gamma n} \times \overline{\alpha+\beta z^{n}+\gamma z^{2n}}|^{q} \times : \frac{1}{t-2}$ $- \frac{\overline{s-2}.\beta A}{\overline{t-3}.\gamma z^{n}} - \frac{\overline{s-3}.\beta B+\overline{r-2}.\alpha A}{\overline{t-4}.\gamma z^{2n}} - \frac{\overline{s-4}.\beta C+\overline{r-3}.\alpha B}{\overline{t-5}.\gamma z^{3n}}$ − &c.

When any particular example is propoſed, if, after having ſubſtituted in the former or aſcending form, it come out an infinite ſeries, let the latter or deſcending form be tried; for ſometimes an area is quadrable by the one ſeries when it is not ſo by the other.

EXAMPLE I.

Let the ordinate y be $= \frac{a}{\sqrt{z^{6}-2z^{7}+3z^{8}}}$; which reduced to form is $az^{-2-1} \times \overline{1-2z+3z^{2}}|^{\frac{1}{2}-1}$.

Here then we have $a=a$; $\alpha=1$, $\beta=-2$, $\gamma=3$; $p=-2$, $n=1$, $q=\frac{1}{2}$. Whence $r=\frac{p}{n}=-2$, $s=r+q$ $=-\frac{2}{3}$, $t=s+q=-1$, $v=t+q=-\frac{1}{2}$, $w=v+q=0$, &c.

And by ſubſtituting theſe values in the aſcending ſeries, we obtain $az^{-2} \times \overline{1-2z+3z^{2}}|^{\frac{1}{2}} \times -\frac{1}{2}-\frac{3z}{2}$ (all the reſt of the terms vaniſhing) $= \frac{1+3z}{2zz} \times -a\sqrt{1-2z+3zz}$ for the area required, and which therefore is quadrable.

But if the ſame values be ſubſtituted in the deſcending ſeries, we get the area expreſſed by the infinite

finite ſeries $\frac{a\sqrt{1-2z+3z^2}}{9z^4} \times : -1 - \frac{7A}{4z} - \frac{9B-4A}{5z^2} - \frac{11C-5B}{6z^3} - \&c. = \frac{a\sqrt{1-2z+3z^2}}{9z^4} \times : -1 + \frac{7}{4z} - \frac{9.7+4.4}{4.5z^2} + \frac{11\times\overline{9.7+4.4}+5.5}{4.5.6z^3} - \&c. = \frac{a\sqrt{1-2z+3z^2}}{9z^4} \times : -1 + \frac{7}{4z} - \frac{79}{4.5z^2} + \frac{894}{4.5.6z^3} - \&c.$

EXAMPLE II.

Suppoſe $y = \frac{azz}{\overline{2+3z+3z^2}\big|^{\frac{2}{3}}}$; which reduced to the proper form is $az^{3-1} \times \overline{2+3z+3z^2}\big|^{\frac{1}{3}-1}$.

Here $a = a$; $\alpha = 2$, $\beta = 3$, $\gamma = 3$, $p = 3$, $n = 1$, $q = \frac{1}{3}$.
Then $r = \frac{p}{n} = 3$, $s = 3\frac{1}{3}$, $t = 3\frac{2}{3}$, $v = 4$, &c.

And, by the aſcending ſeries, the area will be expreſſed by the infinite ſeries $\frac{1}{2}az^3\sqrt[3]{2+3z+3z^2} \times : \frac{1}{3} - \frac{10A}{8}z - \frac{13B+11A}{10}z^2 - \frac{16C+14B}{12}z^3 - \frac{19D+17C}{14}z^4 - \&c. = az^3\sqrt[3]{2+3z+3z^2} \times : \frac{1}{6} - \frac{10}{6.8} + \frac{42}{6.8.10} + \frac{728}{6.8.10.12} - \&c.$ and therefore it is not quadrable by this method.

But by the deſcending ſeries the area is quadrable, and comes out $\frac{z-2}{5} \times a\sqrt{2+3z+3z^2}$, all the terms after the ſecond vaniſhing.

So that ſometimes the area is quadrable by the one ſeries, and ſometimes by the other; but it is alſo ſometimes quadrable by neither, and ſometimes by them both.

Co-

COROLLARY III.

In the caſe of a binomial $y = az^{p-1} \times \overline{\alpha + \beta z^{n}}\big|^{q-1}$, beſide the letters which in the laſt corollary were equal to nothing, γ alſo will be nothing; and by ſuppoſing it to vaniſh in the aſcending ſeries in the laſt corollary, there will reſult $\frac{az^{p}}{\alpha n} \times \overline{\alpha + \beta z^{n}}\big|^{q} \times : \frac{1}{r}$ $- \frac{s\beta}{\overline{r+1}\cdot\alpha} Az^{n} - \frac{\overline{s+1}\cdot\beta}{\overline{r+2}\cdot\alpha} Bz^{2n} - \frac{\overline{s+2}\cdot\beta}{\overline{r+3}\cdot\alpha} Cz^{3n} - \frac{\overline{s+3}\cdot\beta}{\overline{r+4}\cdot\alpha} Dz^{4n}$ – &c. for the aſcending ſeries to be uſed in this caſe; where the ſeveral letters have the ſame value as before. And this ſeries, it is evident, will always terminate when s is either nothing or a negative integer, and the number of terms will be one more than the number of units in s.

But the ſame ordinate $y = az^{p-1} \times \overline{\alpha + \beta z^{n}}\big|^{q-1}$ is = $az^{p-1+qn-n} \times \overline{\beta + \alpha z^{-n}}\big|^{q-1}$. Then by writing, in the above aſcending ſeries, α for β, β for α, $p+qn-n$ for p, and $-n$ for $+n$, we ſhall have $\frac{az^{p-n}}{\beta n} \times \overline{\alpha + \beta z^{n}}\big|^{q}$ $\times : \frac{1}{s-1} - \frac{\overline{r-1}\cdot\alpha A}{\overline{s-2}\cdot\beta z^{n}} - \frac{\overline{r-2}\cdot\alpha B}{\overline{s-3}\cdot\beta z^{2n}} - \frac{\overline{r-3}\cdot\alpha C}{\overline{s-4}\cdot\beta z^{3n}}$ – &c. for the area expreſſed by a deſcending ſeries; and which, it is evident, will terminate and be quadrable when r is any affirmative integer number.

Ex-

EXAMPLE I.

If the ordinate be $y=\frac{azz}{bb+2bcz^3+c^2z^6}=\frac{azz}{\overline{b+cz^3}|^2}$ $azz\times\overline{b+cz^3}|^{-2}=az^{3-1}\times\overline{b+cz^3}|^{-1-1}$.

We ſhall have $a=a$; $\alpha=b$, $\beta=c$; $p=3$, $n=3$, and $q=-1$. Hence $r=\frac{p}{n}=1$, and $s=r+q=1-1=0$; ſo that the curve is quadrable by both forms of the ſeries. Thus,

By ſubſtituting in the aſcending ſeries, we ſhall have $\frac{az^3}{3c}\times\overline{b+cz^3}|^{-1}\times\frac{1}{1}=\frac{az^3}{3b\times\overline{b+cz^3}}$ for the area on the one ſide of the ordinate.

And by uſing the deſcending ſeries, we obtain $\frac{az^0}{3b}\times\overline{b+cz^3}|^{-1}\times-\frac{1}{1}=-\frac{a}{3c\times\overline{b+cz^3}}$ for the area on the other ſide of the ordinate.

EXAMPLE II.

Suppoſe the ordinate be $y=\frac{az^3}{\sqrt{b+cz^2}}=az^3\times$ $\overline{b+cz^2}|^{-\frac{1}{2}}=az^{4-1}\times\overline{b+cz^2}|^{\frac{1}{2}-1}$.

Then $a=a$; $\alpha=b$, $\beta=c$; $p=4$, $n=2$, and $q=\frac{1}{2}$. Hence $r=\frac{p}{n}=2$, and $s=r+q=2\frac{1}{2}$. So that it appears that the deſcending ſeries will bring out a terminate, but the aſcending one an interminate area. Thus, By ſubſtituting in the aſcending ſeries, we have $\frac{az^4\sqrt{b+cz^2}}{2b} \times : \frac{1}{2} - \frac{5cz^2}{2.3.2b} - \frac{5.7c^2z^4}{2.3.4.2^2b^2} - \frac{5.7.9c^3z^6}{2.3.4.5.2^3b^3} - \&c.$ for the area. Or by writing d for $2b$, it will be $\frac{az^4\sqrt{b+cz^2}}{d} \times : \frac{1}{2} - \frac{5cz^2}{2.3d} - \frac{5.7c^2z^4}{2.3.4d^2} - \frac{5.7.9c^3z^6}{2.3.4.5d^3} - \&c.$

But by ſubſtituting in the deſcending ſeries, the area is quadrable, and comes out $\frac{az^2\sqrt{b+czz}}{2c} \times \overline{\frac{2}{3} - \frac{4b}{3czz}} = a\sqrt{b+czz} \times \frac{czz-2b}{3cc}$.

EXAMPLE III.

If the ordinate y be $= \sqrt{a \times \overline{b+x}} = \sqrt{ab+ax} = a^{\frac{1}{2}}x^0 \times \overline{b+x}|^{\frac{1}{2}} = a^{\frac{1}{2}}x^{1-1} \times \overline{b+x}|^{\frac{3}{2}-1}$; which expreſſes the common parabola, a being the parameter, and $b+x$ the whole abſciſſa or diſtance of the ordinate from the vertex of the curve.

Here $a = a^{\frac{1}{2}}$; $\alpha = b$, $\beta = 1$, $z = x$; $p = 1$, $n = 1$, and $q = \frac{3}{2}$. Hence $r = \frac{p}{n} = 1$, and $s = r + q = \frac{5}{2}$. So that the curve is quadrable by the deſcending ſeries, but not by the aſcending one. Thus,

By ſubſtituting in the deſcending ſeries, we obtain $a^{\frac{1}{2}}x^0 \times \overline{b+x}|^{\frac{3}{2}} \times \frac{2}{3} = \frac{2}{3}\sqrt{a} \times \overline{b+x}|^{\frac{3}{2}}$ for the area.——And ſince this area vaniſhes only when $b+x$ or the whole abſciſſa is equal to nothing, it denotes the area of the whole parabola. But when x is $= 0$, this area becomes $\frac{2}{3}\sqrt{a} \times b^{\frac{3}{2}}$ for the area to the abſciſſa b; hence, by correction, $2\sqrt{a} \times \frac{\overline{b+x}|^{\frac{3}{2}} - b^{\frac{3}{2}}}{3}$

will be the area of the fruſtum included by the two ordinates anſwering to the two abſciſſas $b+x$ and b. And, after the ſame manner, $2\sqrt{a}\times\frac{\overline{b+x}|^{\frac{3}{2}}-\overline{b-x}|^{\frac{3}{2}}}{3}$ will be found to denote the fruſtum contained by the two ordinates whoſe abſciſſas are $b+x$ and $b-x$.

But by ſubſtituting in the aſcending ſeries, the area will come out the infinite ſeries $\frac{x\sqrt{a}}{b}\times\overline{b+x}|^{\frac{1}{2}}\times$ $1-\frac{5x}{4b}+\frac{5.7x^2}{4.6b^2}-\frac{5.7.9x^3}{4.6.8b^3}+$ &c.

As this ſeries vaniſhes when x is put equal to nothing, it expreſſes the area beginning where x begins, viz. the fruſtum included by the two ordinates whoſe abſciſſas are b and $b+x$; and therefore this ſeries is equal to $2\sqrt{a}\times\frac{\overline{b+x}|^{\frac{3}{2}}-b^{\frac{3}{2}}}{3}$, which was found above to expreſs the ſame area. Then, by making this quantity equal to the ſaid ſeries, and reducing, we obtain $\frac{2b}{3x}\times\overline{1-\frac{b}{b-x}}|^{\frac{3}{2}}$ for the value or ſum of the infinite ſeries $1-\frac{5x}{4b}+\frac{5.7x^2}{4.6b^2}-\frac{5.7.9x^3}{4.6.8b^3}$ + &c. And by ſuppoſing b and x in theſe laſt expreſſions to be equal to each other, there will reſult $\frac{2\sqrt{2}-1}{3\sqrt{2}}$ for the ſum of the infinite ſeries $1-\frac{5}{4}+\frac{5.7}{4.6}-\frac{5.7.9}{4.6.8}-\frac{5.7.9.11}{4.6.8.10}-$ &c. or $\frac{1+\sqrt{2}}{3\sqrt{2}}=\frac{5}{4}-\frac{5.7}{4.6}+\frac{5.7.9}{4.6.8}-\frac{5.7.9.11}{4.6.8.10}+$ &c.

Now ſuch kinds of diverging ſeries have generally been thought to be of an infinite magnitude, and their ſums abſolutely unaſſignable; but that there is nothing abſurd in aſſerting the foregoing ſeries, and conſequently any other of the ſame kind, to have a determinate finite ſum, may be eaſily made evident thus: Notwithſtanding any term of this ſeries doth exceed any of the preceding terms, yet it is evident that it exceeds the term immediately preceding it by a leſs quantity than *that* by which this laſt term exceeds *its* preceding one, and therefore the differences of every two adjacent terms being taken, thoſe differences will form an infinite converging ſeries, of which the firſt or greateſt term will be the difference between the firſt two adjacent terms of the original ſeries, and conſequently the ſum of this ſeries of differences will be of a finite magnitude; but the ſum of this ſeries is equal to the ſum of the original ſeries, and therefore that alſo hath a finite ſum.

COROLLARY IV.

In the caſe of a ſimple nominal $y = az^{p-1} \times \alpha^{q-1} = a\alpha^{q-1} z^{p-1}$, ſuppoſing β in the laſt corollary to become nothing, the area will be $\frac{a\alpha^{q-1} z^{p}}{p} = \frac{mz^{p}}{p}$, putting $m = a\alpha^{q-1}$.

Or if inſtead of mz^{p-1} be written its value y, the area will be $\frac{yz}{p}$.

And the curve in this caſe is always quadrable except when p is $= 0$; for then the area $\frac{mz^{p}}{p}$ becomes $\frac{mz^{0}}{0} = \frac{m}{0}$, viz. infinitely great.

To this caſe of areas belong all ſimple parabolic ſpaces, and the ſpaces lying between the curves and aſymptotes of all hyperbolas. And all ſuch are, evidently, to the parallelograms of the ſame baſe and altitude as 1 to p.

EXAMPLE I.

The general equation to parabolas is $p^m z^n = y^{m+n}$, or $y = p^{\frac{m}{m+n}} z^{\frac{n}{m+n}}$; where y is the ordinate, z the abſciſſa, and p a given quantity.

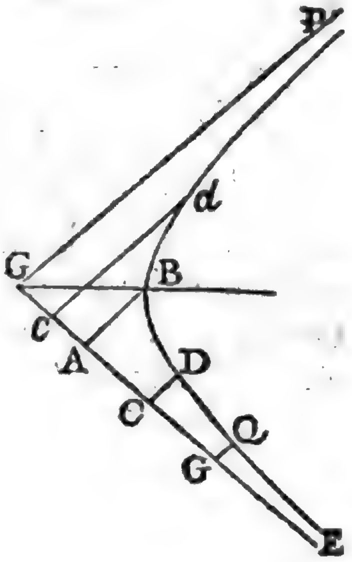
Here then $m = p^{\frac{m}{m+n}}$, and $p = \frac{n}{m+n} + 1 = \frac{m+2n}{m+n}$; which value of p being of ſome magnitude, we conclude that all parabolas are quadrable, and the area will be, by ſubſtituting in the two forms above, either $\frac{m+n}{m+2n} \times p^{\frac{m}{m+n}} z^{\frac{m+2n}{m+n}}$, or $\frac{m+n}{m+2n} \times yz$.

So that the area of any parabola is to that of its circumſcribed parallelogram, as $m+n$ is to $m+2n$.

In the common parabola m and n are each equal to 1, or $pz = yy$, and the general area becomes $\frac{2}{3}yz$ or $\frac{2}{3}$ of the circumſcribed parallelogram.

EXAMPLE II.

The general equation to hyperbolas is $z^m y^n = p^{m+n}$, or $y = p^{\frac{m+n}{n}} z^{\frac{-m}{n}}$; where z is = one aſymptote GC, and y = an ordinate CD drawn parallel to the other GP.



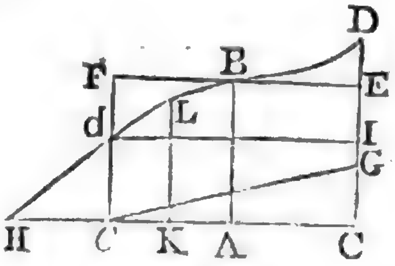
Here then $m = p^{\frac{m+n}{n}}$, and $p = 1 - \frac{m}{n} = \frac{n-m}{n}$; ſo that the ſpace will always be quadrable unleſs when n and m are equal to each other, as in the caſe of the conic parabola; and by ſubſtituting in the two foregoing forms, we obtain either $\frac{np^{\frac{n+m}{n}} z^{\frac{n+m}{n}}}{n-m}$ or $\frac{nyz}{n-m}$ for the area $GPdDC$.

So that hyperbolic ſpaces are to their inſcribed parallelograms as $\frac{n}{n-m}$ to 1, or as n to $n-m$; and ſince when n is $= m$, as in the conic parabola, this ratio becomes that of n to 0; it appears that the ſaid hyperbolic ſpace $GPdDC$ is in this caſe infinitely great; but in any other, of a finite magnitude, notwithſtanding it is of an infinite length towards P.

COROLLARY V.

If in the original ſeries, in the propoſition, only β, γ, δ, &c. vaniſh, by becoming equal to nothing, the ſeries for the area will become $\alpha^{q-1} z^{p} \times : \frac{a}{p} + \frac{bz^{n}}{p+n} + \frac{cz^{2n}}{p+2n} + \frac{cz^{3n}}{p+3n} +$ &c.

And if here again p and n be ſuppoſed each equal to 1, and α^{q-1} be actually multiplied into the ſeries, and then the literal coefficients of the ſeveral terms be repreſented by A, B, C, &c. we ſhall obtain $Ax + \frac{1}{2}Bx^{2} + \frac{1}{3}Cx^{3} + \frac{1}{4}Dx^{4} +$ &c. for the area correſponding to the ordinate $y = A + Bx + Cx^2 + Dx^3 +$ &c. putting x for z.



And by this ſeries we might find a great variety of areas; but to reduce it to another ſeries converging quicker; ſince, if x be $= AC$, $y = CD$, the above ſeries will repreſent the area $ABDC$ of the curve dBD; ſo likewiſe if y flow back from A to cd, making the ordinate $y = A - Bx + Cx^2 + Dx^3 +$ &c. x being now negative, the area $ABdc$ will be $Ax - \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 - \frac{1}{4}Dx^4 +$ &c.

Then, ſuppoſing the former x or AC to be equal to this latter x or Ac, and adding this ſeries to the former one, there will reſult $2Ax + \frac{2}{3}Cx^3 + \frac{2}{5}Ex^5 +$ &c. for the area $cdDC$. Or if now z be put $= cC = 2x$, this area will be equal to $Az + \frac{Cz^3}{3\cdot 4} + \frac{Ez^5}{5\cdot 4^2} + \frac{Gz^7}{7\cdot 4^3} +$ &c. where c may either be at the vertex of the curve or not.

When x is $= 0$, then $y = A = AB$, ſo that AB is always the value of A, or A denotes the middle ordinate of the area $cdDC$.

If FBE be drawn parallel to cC, and meeting CD and cd in E and F; the rectangle $cFEC$ will be equal to $cC \times AB = Az$, and conſequently the difference between $cdDC$ and $cFEC$ will be $\frac{Cz^3}{3\cdot 4} + \frac{Ez_5}{3\cdot 4^2} +$ &c.

EXAMPLE I.

If dBD be a right line, its equation will be $y = A + Bx$; B here denoting the ratio of the ſine of the angle dBA to its coſine. Then, ſince the value of each of the letters C, D, E, &c. in the general formula is nothing, the general area $Az + \frac{Cz^3}{3.4} + \&c.$ will be only $Az =$ the rectangle $CF = AB \times cC = \frac{dc + DC}{2} \times cC$ for the area $dDCc$, as in rule 6 prob. 3 ſect. 1 part 2.

EXAMPLE II.

If dBD be a parabola, its equation will be $y = \sqrt{a \times \overline{b+x}} =$ (by extracting the root) $\sqrt{ab} \times : 1 + \frac{x}{2b} - \frac{x^2}{2.4b^2} + \frac{1.3x^3}{2.4.6b^3} - \frac{1.3.5x^5}{2.4.6.8b^4} + \&c.$

Here $A = \sqrt{ab}$, $C = -\frac{A}{2.4b^2}$, $E = -\frac{1.3.5A}{2.4.6.8b^4}$, &c. and, by ſubſtituting, the area $cdDC$ will be $= 2A$ (viz. $2\sqrt{ab}$) $\times : x - \frac{x^3}{4.6b^2} - \frac{1.3.5x^5}{4.6.8.10b^4} - \frac{1.3.5.7.9x^7}{4.6.8.10.12.14b^6} - \&c. = Az \times : 1 - \frac{z^2}{4.6.2^2b^2} - \frac{1.3.5z^4}{4.6.8.10.2^4b^4} - \&c.$ And therefore the rectangle Az or $BA \times cC$ exceeds the parabolic area by all the terms of the ſeries after the firſt.

But by example 3 to corollary 3, the ſame area $cdDC$ is $= 2\sqrt{a} \times \frac{\overline{b+x}|^{\frac{3}{2}} - \overline{b-x}|^{\frac{3}{2}}}{3}$; conſequently $\frac{\overline{b+x}|^{\frac{3}{2}} - \overline{b-x}|^{\frac{3}{2}}}{3}$ is $= \sqrt{b} \times : x - \frac{x^3}{4.6b^2} - \frac{1.3.5x^5}{4.6.8.10b^4} - \frac{1.3.5.7.9x^7}{4.6.8.10.12.14b^6} - \&c.$ And when b and x are equal to each other in this equation, we obtain $\frac{3 - 2\sqrt{2}}{3}$ for the value of the infinite ſeries $\frac{1}{4.6} + \frac{1.3.5}{4.6.8.10} + \frac{1.3.5.7.9}{4.6.8.10.12.14} + \&c.$

When c is the vertex of the curve, then b is $= x$, and the ſeries for the area cDC becomes $Az \times : 1 - \frac{1}{4.6} - \frac{1.3.5}{4.6.8.10} -$ &c. which alſo, by ſubſtituting for this laſt ſeries its value $1 - \frac{3 - 2\sqrt{2}}{3} = \frac{2\sqrt{2}}{3}$ as found above, will be $= \frac{2\sqrt{2}}{3} Az = \frac{2\sqrt{2}}{3} \times$ the rectangle $cFEC$.

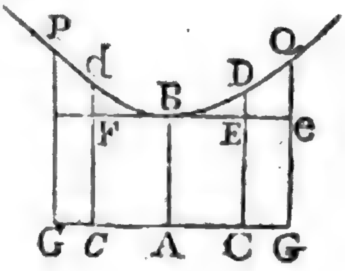
EXAMPLE III.

If the curve be a circle, and A were any point between the circumference and the center, the ſeries would be very complex; but if A be the center, and r the radius, then will y be $= \sqrt{rr - xx} = r \times : 1 - \frac{x^2}{2r^2} - \frac{x^4}{2.4r^4} - \frac{1.3x^6}{2.4.6r^6} -$ &c. and conſequently the area will be $2rx \times : 1 - \frac{x^2}{2.3r^2} - \frac{x^4}{2.4.5r^4} - \frac{1.3x^6}{2.4.6.7r^6} - \frac{1.3.5x^8}{2.4.6.8.9r^8} -$ &c. which ſeries converges very faſt when x is ſmall in compariſon with r.

When x becomes equal to r, we obtain $4rr$ (or the ſquare of the diameter) $\times : 1 - \frac{1}{2.3} - \frac{1}{2.4.5} - \frac{1.3}{2.4.6.7} - \frac{1.3.5}{2.4.6.8.9} -$ &c. for the whole circle.

Hence $1 - \cdot 785398$ &c. is $= \frac{1}{2.3} + \frac{1}{2.4.5} + \frac{1.3}{2.4.6.7} +$ &c.

EXAMPLE IV.



If dBD be an hyperbola; put its ſemi-tranſverſe $AB = t$, ſemi-conjugate $AG = c$, abſciſſa $AC = x$, and ordinate $CD = y$; then its equation will be $yy = tt \times \frac{cc + xx}{cc}$, or

$y = t\sqrt{1 + \frac{xx}{cc}} = t \times : 1 + \frac{x^2}{2c^2} - \frac{x^4}{2.4c^4} + \frac{1.3x^6}{2.4.6c^6} - \frac{1.3.5x^8}{2.4.6.8c^8}$ + &c. and, by ſubſtituting in the general ſeries, the area $cdDC$ will be $= 2tx \times : 1 + \frac{x^2}{2.3c^2} - \frac{x^4}{2.4.5c^4} + \frac{1.3x^6}{2.4.6.7c^6}$ − &c. the ſame with the ſeries in the laſt example for the area of a circle, differing only in the ſigns of the ſecond, fourth, ſixth, &c. terms.

When x is $= c$, the ſeries becomes $2tc$ (or $AB \times GG$) $\times : 1 + \frac{1}{2.3} - \frac{1}{2.4.5} + \frac{1.3}{2.4.6.7}$ − &c. for the area $GPQG$.

The rectangle $ABEC$ is $= tx$, which taken from the area $ABDC$ will leave $AB \times BE \times : \frac{x^2}{2.3c^2} - \frac{x^4}{2.4.5c^4} + \frac{1.3x^6}{2.4.6.7c^6}$ − &c. for the area BDE. So that the rectangle $ABEC$ is to the trilineal BDE, as 1 is to the ſeries $\frac{x^2}{2.3c^2} - \frac{x^4}{2.4.5c^4} + \frac{1.3x^6}{2.4.6.7c^6}$ − &c. And when x or AC is $= c$ or AG, the rectangle $ABeG$ will be to the trilineal BeQ, as 1 is to $\frac{1}{2.3} - \frac{1}{2.4.5} + \frac{1.3}{2.4.6.7}$ − &c.

Parallel to one aſymptote GP (figure to example 2 cor. 4) of an hyperbola draw ordinates BA, CD, to the other GG; B being the vertex of the curve: And put $GA = a$, $AB = b$, $AC = x$, and $CD = y$.

Then $a + x : a :: b : y = \frac{ab}{a+x} = b \times : 1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3}$ +− &c. = the ordinate.

And by comparing this with the general ſeries, we obtain $2bx \times : 1 + \frac{x^2}{3a^2} + \frac{x^4}{5a^4} + \frac{x^6}{7a^6}$ + &c. for the area $cdDC$, taking $Ac = AC$.

And when $x = a$, or $AC = AG$, the above expreſſion will become $2GA \times AB \times : 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} +$ &c. for the area $GPBQG$.

PROPOSITION II.

If there be any ſolid, in which all the parallel ſections, either right or oblique to the axe, are like and ſimilar figures, and if, when the abſciſſa, or part of the axe drawn through the centers of the parallel ſections, is repreſented by z, *the value of the ſection be expreſſed by any ſeries of terms involving* z *and known quantities, after the ſame manner as is expreſſed the ordinate in the laſt propoſition; then will the ſolidity be expreſſed by the ſame quantity as the area in the laſt propoſition: That is, ſuppoſing the relation between the abſciſſa and ſection to be expreſſed by any equation of which one ſide is the ſection, after the manner of the relation between the abſciſſa and ordinate in the laſt propoſition; whenever the value of the ordinate agrees with that of the ſection, then will the value of the ſolidity be the ſame with that of the area as found by the ſaid propoſition.*

DEMONSTRATION.

For, the fluxion of the ſolid being equal to the ſection drawn into the fluxion of the abſciſſa, and the fluxion of the area equal to the ordinate drawn into the ſame fluxion of the abſciſſa, whenever the ordinate and ſection agree, the area and ſolidity muſt likewiſe agree, ſince equal fluxions have equal fluents. *Q.E.D.*

SCHOLIUM.

All the examples that have been given to the corollaries to the laſt propoſition may be conſidered as examples to this, the quadrature in thoſe being conſidered as the cubature here; and it is evident that curves will be cubable not only whenever they are quadrable, but will often be cubable when the area cannot be expreſſed but in an infinite ſeries, becauſe that the ſection of a ſolid (uſing the area of a given circle as a given number) is often a terminate expreſſion when the ordinate is denoted by an infinite ſeries; and this is the caſe in the conic ſections; for thoſe curves (the parabola excepted) are not duadrable, yet all the ſolids generated by their revolution have finite expreſſions, as we have already found in the foregoing part of this work, and as will more generally appear in what follows; which I have put down not merely to ſhew that the ſame concluſions may be brought out by different means, but chiefly for the ſake of ſome eaſy, curious, and general rules with which this method of inveſtigation ſo eaſily ſupplies us.

COROLLARY.

Thus, by applying here what is done in the laſt corollary of the laſt propoſition to the curves of the ſecond kind, we ſhall obtain general and terminate expreſſions for their ſolidities. For, by prop. 2 ſect. 2 part 3, the flowing ſection being always as $A + Bx + Cxx$, if p denote a given quantity, $p \times \overline{A + Bx + Cxx}$ will denote the ſection itſelf;

and conſequently $px \times \overline{A + \frac{1}{2}Bx + \frac{1}{3}Cx^2}$ or $p \times \overline{Az + \frac{1}{12}Cz^3}$ will denote the ſolidity $cdDC$ generated by the ſection of a ſolid in flowing from the place cd to the place CD, z being $= cC$. [Fig. to corollary 5 to the laſt prop.

Since pA repreſents the middle ſection AB, the firſt term pAz in the above expreſſion will be equal to the cylinder $cFBEC$ whoſe baſe is equal to the middle ſection AB, and altitude $cC = z$; and conſequently $\frac{1}{12}pCz^3$ will always be the difference between the ſolid $cdBDC$ and the cylinder $cFBEC$.

Or if there be taken $CG : Cc :: \sqrt{C} : 1$, and the three areas, viz. the curve $cdBDC$, the rectangle $cFBEC$, and the right-angled triangle cGC, be ſuppoſed to revolve together about the common axe cC; then will the difference between the ſolid generated by $cdBDC$ and the cylinder generated by $cFBEC$ be conſtantly equal to one-fourth of the cone generated by cGC.

The fruſt. generated by $cdBDC$ is $\left\{\begin{array}{l}\text{greater than}\\ \text{equal to}\\ \text{or leſs than}\end{array}\right\}$ the cylinder generated by $cFBEC$ according as the value of C is $\left\{\begin{array}{l}\text{affirmative}\\ \text{nothing}\\ \text{or negative}\end{array}\right\}$, that is, according as the curve dBD is $\left\{\begin{array}{l}\text{an hyperbola}\\ \text{a parabola}\\ \text{or an ellipſe}\end{array}\right\}$.

In the parabola C is $= 0$, the equation to the curve being $yy = A + Bx$; and therefore the parabolic conoid, or any fruſtum of it, is equal to a cylinder of the ſame height with it, and whoſe end is equal to the middle ſection of the conoid, viz. the ſection parallel to and equally diſtant from its two ends.

In the hyperbola $\surd C$ is $= \frac{n}{m}$, putting m for the whole axe of which cC is the continuation, and n for its conjugate axe; and therefore the hyperbolic conoid, or its fruſtum, exceeds the cylinder by one-fourth of a cone of which the radius of the baſe is to the common altitude (cC of all the three ſolids) as n is to m.

But in the ellipſe the value of C is $-\frac{nn}{mm}$, m and n being the axes as above; and therefore the ſemi-ſpheroid, or its fruſtum, is leſs than the cylinder by one-fourth of the ſaid cone of which the radius of the baſe is to the common altitude as n is to m.

When the ellipſe becomes a circle, m will be equal to n; and conſequently the ſemi-ſphere, or its fruſtum, will be leſs than the cylinder by one-fourth of the cone of which the radius of the baſe is equal to the common altitude of all the ſolids.

When dBD is a right line, or the ſolid a cone, then the value of $\surd C$ or $\frac{GC}{Cc}$ will be equal to $\frac{DC}{CH}$, producing the right lines Dd, Cc to meet in H; conſequently the triangles DHC, GcC are ſimilar, and the conic fruſtum exceeds the cylinder by one-fourth of a ſimilar cone, the ſolids being ſtill all of the ſame altitude. The radius CG of the baſe of the ſaid ſimilar cone will evidently be $= \frac{DC \times Cc}{CH}$ a fourth proportional to HC, CD and Cc: Or, if dI

be drawn parallel to cC, the triangle dID will be equal to the triangle cCG in all reſpects, or $CG = ID = DC - cd$, that is, the radius CG of the baſe of the ſimilar cone is equal to the difference between the greateſt CD and leaſt cd radius of the fruſtum.——When the fruſtum becomes a compleat cone, it is evident that it will exceed the cylinder by one-fourth of itſelf, or the cone will be to the cylinder as 4 is to 3.

From the general manner of conſidering the generation of the fruſtum, in the beginning of this corollary, by the parallel motion of a flowing ſection, it is evident that the above properties will obtain in the ſame ſolids whether the ends are perpendicular or oblique to the axe; and alſo that the general method will include the fruſtum of any pyramid whether right or oblique; and ſuch a fruſtum will exceed the priſm, of the ſame altitude and whoſe baſe is equal to the middle ſection of the fruſtum, by one-fourth of a cone of the ſame altitude alſo and of which the radius of the baſe is $\frac{DC \times Cc}{CH}$.

It may farther be obſerved in general that in the ſame or in ſimilar ſolids, when the altitude and the inclination of the ends to the axe are the ſame, the cone, or difference between the fruſtum of the ſolid and the cylinder or priſm, will be conſtantly the ſame quantity, whatever the magnitude of the ends may be. When the altitude is conſtant, and the inclinations of the ends vary, the ſaid difference is reciprocally as the cube of the diameter of the generating plane which is conjugate to *Cc* the axe of the fruſtum or diameter connecting the centers of its ends. When both the altitude and inclination vary, the difference is as the cube of the altitude directly and cube of the ſaid conjugate diameter reciprocally; but when they vary ſo as that the altitude is always reciprocally as that diameter, then the difference is a conſtant quantity.

PROPOSITION III.

In the fruſtum of any ſolid generated by the revolution of any conic ſection about its axe, if to the ſum of the two ends be added four times the middle ſection or ſection parallel to and equally diſtant from the two ends, one-ſixth of the laſt ſum will be a mean area, and being drawn into the altitude of the ſolid, the product will be equal to the content.

DEMONSTRATION.

By the corollary to the laſt prop. the content is $px \times \overline{A + \frac{1}{2}Bx + \frac{1}{3}Cx^2} = px \times \frac{6A + 3Bx + 2Cx^2}{6}$; but by prop. 2 ſect. 2 part 3, the one end pD^2 is $= p \times \overline{A + Bx + Cx^2}$, then by writing $\frac{1}{2}x$ for x the middle ſection will be $p\delta^2 = p \times \overline{A + \frac{1}{2}Bx + \frac{1}{4}Cx^2}$, and by writing 0 for x the other end will be $pd^2 = pA$; now it is evident that the ſum of the two ends with four times the middle ſection is equal to the numerator of the quantity expreſſing the content; conſequently $px \times \frac{D^2 + 4\delta^2 + d^2}{6} \left(= px \times \frac{6A + 3Bx + 2Cx^2}{6}\right)$ will be equal to the ſolidity of the fruſtum. *Q.E.D.*

Co-

COROLLARY I.

When the fruſtum becomes the compleat ſolid, the leſs end vaniſhes, and the content is barely $px \times \frac{D^2 + 4\delta^2}{6}$.

COROLLARY II.

This propoſition, it is evident, includes all fruſtums (as well as the compleat ſolids) whether right or oblique, not only of the ſolids generated from the revolution of the conic ſections, but alſo of all pyramids, cones, and in ſhort of any ſolid whoſe parallel ſections are ſimilar figures. The ſame theorem may alſo be applied to the areas of all curves whoſe equation is of this form $y = A + Bx + Cx^2$, calling D and d the extreme ordinates, and δ the middle one.

And of this form is the equation to the parabola dBD (figure to corollary 5 to prop. 1) CD being the axe of the curve, and putting $cA = x$, and $AB = y$; and conſequently any parabolic portion $cdBA$ bounded by the curve dB, the two lines AB, cd parallel to the axe CD, and the line Ac perpendicular to the ſame, is truly expreſſed by $\frac{AB + 4KL + cd}{6} \times Ac$, KL being the middle ordinate.

SECTION II.

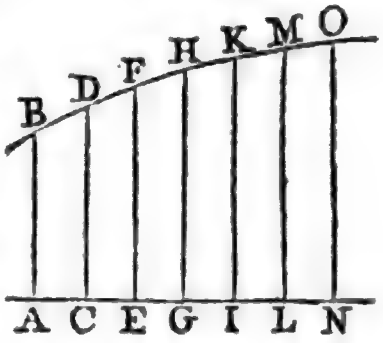
Of the approximate QUADRATURE *and* CUBATURE *of* CURVES *by means of* EQUIDISTANT ORDINATES *or* SECTIONS.

PROPOSITION I.

IF any right line AN *be divided into any even number of equal parts* AC, CE, EG, *&c. and at the points of diviſion be erected perpendicular ordinates* AB, CD, EF, *&c. terminated by any curve* BDF *&c. and if* A *be put for the ſum of the extreme or firſt and laſt ordinates* AB, NO; B *for the ſum of the even ordinates* CD, GH, LM, *&c. viz. the ſecond, fourth, ſixth, &c. and* C *for the ſum of all the reſt* EF, IK, *&c. viz. the third, fifth, &c. or the odd ordinates wanting the firſt and laſt: Then I ſay that the common diſtance* AC *or* CE, *&c. of the ordinates being drawn into the ſum ariſing from the addition of* A, 4 *times* B, *and* 2 *times* C, *one-third of the product will be the area* ABON *very nearly.*

That is, $\frac{A+4B+2C}{3} \times D =$ *the area, putting* D = AC *the common diſtance of the ordinates.*

DEMONSTRATION.



If through the firſt three points *B*, *D*, *F*, of the propoſed curve a parabola be conceived to be drawn, having its axe parallel to the ordinates; the parabolic area, by the ſecond corollary of the laſt propoſition will be $\frac{AB+4CD+EF}{6} \times AE$ $= \frac{AB+4CD+EF}{3} \times AC$; but when the points B, D, F, are at no great diſtance from each other, the parabolic curve will nearly coincide with the curve propoſed, and conſequently the area of the one equal to that of the other very nearly; hence $\frac{AB+4CD+EF}{3} \times AC$ will be equal to the area $ABFE$ very nearly. After the ſame manner $\frac{EF+4GH+IK}{3} \times AC$ or (EG) will be equal to the area $EFKI$; and $\frac{IK+4LM+NO}{3} \times AC =$ the area $IKON$; and ſo on.

Wherefore, by taking the ſum of theſe areas, we ſhall have $\frac{AB+4CD+2EF+4GH+2IK+4LM+NO}{3}$ $\times AC = \frac{A+4B+2C}{3} \times AC$ for the whole area $ABON$ very nearly. *Q.E.D.*

COROLLARY.

The ſame theorem will alſo obtain for the contents of all ſolids, by uſing the ſections perpendicular to the axe inſtead of the ordinates, as will appear from the laſt propoſition of the laſt ſection; for the fruſtum of the ſolid there may be ſuppoſed to coincide in the extremes and middle with any other fruſtum, after the ſame manner as the parabola with any other curve. The propoſition is accurately true for all parabolic and right-lined areas, as alſo for all ſolids generated from the revolution of conic ſections or right lines, with all kinds of pyramids;

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ramids; and for all other kinds of areas and ſolidities it is a very good approximation.

SCHOLIUM.

It is very evident that the greater the number of ordinates or ſections are uſed, the more accurately will the area or ſolidity be determined; but in a caſe of real practice, ſuch as caſk-gauging, three ſections are ſufficient; and becauſe of the ſimplicity and accuracy of this method, it is by far the moſt proper for practical gaugers of any thing that can well be deviſed; for by only taking the bung and head diameters and a diameter in the middle between them, the ſum of the bung, head, and 4 times the middle circle drawn into half the length of the caſk will be ſix times the content extremely near. And the ſame method may be uſed to good purpoſe in all caſes of ullaging either ſtanding or lying caſks, by taking the area at the top, bottom, and middle of the liquor.

I ſhall ſubjoin two or three examples to ſhew the eaſe and accuracy of the method, referring thoſe who would ſee it more fully illuſtrated by examples &c. to *Mr Moſs's Gauging*.

EXAMPLE I.

Let it be required to find the area of the quadrant of a circle whoſe radius is 1.

Let the radius be divided into 10 equal parts by 11 ordinates, the firſt being nothing and the laſt equal to the radius, and the intermediate 9 ordinates the ſines of the arcs whoſe verſed ſines are reſpectively $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, $\frac{4}{10}$, $\frac{5}{10}$, $\frac{6}{10}$, $\frac{7}{10}$, $\frac{8}{10}$, and $\frac{9}{10}$.

Now, by the property of the circle, we ſhall have
$\sqrt{1{\cdot}9 \times {\cdot}1} = \sqrt{{\cdot}19} = {\cdot}4358899$ = the 2d ordinate, or the firſt of the aforeſaid nine,
$\sqrt{1{\cdot}8 \times {\cdot}2} = \sqrt{{\cdot}36} = {\cdot}6$ = the 3d ordinate,
$\sqrt{1{\cdot}7 \times {\cdot}3} = \sqrt{{\cdot}51} = {\cdot}7141428$ = the 4th,
$\sqrt{1{\cdot}6 \times {\cdot}4} = \sqrt{{\cdot}64} = {\cdot}8$ = the 5th,
$\sqrt{1{\cdot}5 \times {\cdot}5} = \sqrt{{\cdot}75} = {\cdot}8660254$ = the 6th,
$\sqrt{1{\cdot}4 \times {\cdot}6} = \sqrt{{\cdot}84} = {\cdot}9165151$ = the 7th,
$\sqrt{1{\cdot}3 \times {\cdot}7} = \sqrt{{\cdot}91} = {\cdot}9539392$ = the 8th,
$\sqrt{1{\cdot}2 \times {\cdot}8} = \sqrt{{\cdot}96} = {\cdot}9797959$ = the 9th,
$\sqrt{1{\cdot}1 \times {\cdot}9} = \sqrt{{\cdot}99} = {\cdot}9949874$ = the 10th,
the 11th, being = 1 the radius, as was before obſerved.

Hence $A = 0 + 1 = 1$, B = the 2d + 4th + 6th + 8th + 10th = $3{\cdot}9649847$, and C = the 3d + 5th + 7th + 9th = $3{\cdot}296311$, and $D = {\cdot}1$.

Conſequently $\frac{A + 4B + 2C}{3} \times D = \frac{23{\cdot}45256}{3} \times {\cdot}1 = {\cdot}78175$ = the area pretty near the truth.

This area differs from the truth by more than in curves in general, on account of the obliquity of the curve to the ordinates near the beginning of the arc; but if by the ſame rule we find the arc belonging to the 9 greateſt ordinates, which will be ·70363, and to it add the area of the ſemiſegment whoſe altitude is ·2 and baſe ·6, conſidering it as a parabola, viz. $\cdot6 \times \cdot2 \times \frac{2}{3} = \cdot08$, the ſum ·7836 will be nearer the true area; and if ·08175 the true area of the ſegment had been added, we ſhould have had ·78538 the area very exact.

EXAMPLE II.

Taking example 1 to prob. 5 of the parabola,* in which the abſciſſa of a parabola is 2, and the baſe or double ordinate 12, to find the area of the parabola.

Here by taking three ordinates, of which the firſt and laſt are each nothing, and the middle one $= 2$, their common diſtance being 6; we ſhall have $A = 0$, $B = 2$, $C = 0$, and $D = 6$.

Conſequently $\frac{A + 4B + 2C}{3} \times D = \frac{8}{3} \times 6 = 16 =$ the true area as before.

EXAMPLE III.

Given the lengths of five equidiſtant ordinates of an area, or ſections of a ſolid, 10, 11, 14, 16, 16, and the length of the whole baſe 20; to find the area or ſolidity.

Here $A = 10 + 16 = 26$, $B = 11 + 16 = 27$, $C = 14$, and $D = \frac{20}{4} = 5$.

Wherefore $\frac{A + 4B + 2C}{3} \times D = \frac{26 + 108 + 28}{3} \times 5 = \frac{162}{3} \times 5 = 54 \times 5 = 270 =$ the area or ſolidity required.

EXAMPLE IV.

Given eleven ordinates to an hyperbola between the aſymptotes, to find the area. Thus, taking the equation at the end of example 4 to corollary 5 to prop. 1 ſect. 1, in which any ordinate y is $= \frac{ab}{a+x} = \frac{1}{1+x}$; then ſuppoſing the firſt value of x to be nothing, and the laſt value equal to 1, and conſequently the common diſtance of the ordinates $\cdot 1$ or $\frac{1}{10}$; we ſhall have for the ordinates $\frac{1}{1+0}$, $\frac{1}{1+\cdot 1}$, $\frac{1}{1+\cdot 2}$, &c. or $\frac{10}{10}$, $\frac{10}{11}$, $\frac{10}{12}$, $\frac{10}{13}$, $\frac{10}{14}$, $\frac{10}{15}$, $\frac{10}{16}$, $\frac{10}{17}$, $\frac{10}{18}$, $\frac{10}{19}$ and $\frac{10}{20}$.

Here then $A = \frac{10}{10} + \frac{10}{20} = 1 + \frac{1}{2} = 1{\cdot}5$, $B = \frac{10}{11} + \frac{10}{13} + \frac{10}{15} + \frac{10}{17} + \frac{10}{19} = 3{\cdot}4595393$, $C = \frac{10}{12} + \frac{10}{14} + \frac{10}{16} + \frac{10}{18} = 2{\cdot}72817455\frac{5}{6}$, and $D = {\cdot}1$.

Wherefore $\frac{A + 4B + 2C}{3} \times D = {\cdot}69315021 =$ the area required.

PROPOSITION II.

Given any equidiſtant ordinates of a curve, or ſections of a ſolid, as a, b, c, d, e, f, *&c. to find nearly a general expreſſion of the area or ſolidity by the terms* a, b, c, *&c. and the baſe upon which they inſiſt.*

INVESTIGATION.

The firſt differences of the terms a, b, c, d, e, f, &c. are

$$b - a,\ c - b,\ d - c,\ e - d,\ f - e, \text{ \&c.}$$

The dif. of theſe are

$$c - 2b + a,\ d - 2c + b,\ e - 2d + c,\ f - 2e + d, \text{ \&c.}$$

The dif. of theſe are

$$d - 3c + 3b - a,\ e - 3d + 3c - b,\ f - 3e + 3d - c, \text{ \&c.}$$

The dif. of theſe are

$$e - 4d + 6c - 4b + a,\ f - 4e + 6d - 4c + b, \text{ \&c.}$$

The dif. of theſe are

$$f - 5e + 10d - 10c + 5b - a, \text{ \&c.}$$

Then by putting α, β, γ, δ, ϵ, &c. for the firſt term of theſe differences reſpectively, and tranſpoſing, we ſhall have

$b =$

$$\begin{array}{l}
b = +a+\alpha \\
c = -a+2b+\quad\beta \\
d = +a-3b+\ 3c+\quad\gamma \\
e = -a+4b-\ 6c+\ 4d+\ \delta \\
f = +a-5b+10c-10d+5e+\iota \\
\&c. \qquad \&c.
\end{array}$$

Hence, by ſubſtituting the value of b in that of c; and the values of b and c in that of d; and thoſe of b, c, and d in that of e; and thoſe of b, c, d, and e in that of f; and ſo on; we ſhall obtain

$$\begin{array}{l}
a = a \\
b = a+\alpha \\
c = a+2\alpha+\quad\beta \\
d = a+3\alpha+\ 3\beta+\quad\gamma \\
e = a+4\alpha+\ 6\beta+\ 4\gamma+\ \delta \\
f = a+5\alpha+10\beta+10\gamma+5\delta+\iota \\
\&c. \qquad \&c.
\end{array}$$

Where it is evident that the coefficients in the value of any term are the ſame with thoſe of a binomial raiſed to the power denoted by the place or number of the term, and conſequently the x term z will be $a+x\alpha+x.\frac{x-1}{3}\beta+x.\frac{x-1}{2}.\frac{x-2}{3}\gamma+x.\frac{x-1}{2}.\frac{x-2}{3}.\frac{x-3}{4}\delta$ + &c. Which, by actually multiplying the factors together, &c. becomes $z=a$

$$\begin{array}{llll}
+ & \alpha x+ & \frac{1}{2}\beta x^2+ & \frac{1}{2.3}\gamma x^3+\&c. \\
- & \frac{1}{2}\beta x- & \frac{1+2}{2.3}\gamma x^2- & \frac{1+2+3}{2.3.4}\delta x^3-\&c. \\
+ & \frac{1.2}{2.3}\gamma x+ & \frac{1.2+3.\overline{1+2}}{2.3.4}\delta x^2+ & \frac{1.1+3.\overline{1+2}+4.\overline{1+2+3}}{2.3.4.5}\iota x^3+\&c. \\
- & \frac{1.2.3}{2.3.4}\delta x- & \frac{1.2.3+4\times\overline{1.2+3.\overline{1+2}}}{2.3.4.5}\iota x^2 & \&c. \\
+ & \frac{1.2.3.4}{2.3.4.5}\iota x & \&c. & \\
 & \&c. & &
\end{array}$$

or

$$\begin{array}{l}
\text{or } z=a+\alpha x+\frac{1}{2}\beta x^2+\frac{1}{6}\gamma x^3+\frac{1}{24}\delta x^4+\frac{1}{120}\epsilon x^5 \text{ \&c.}\\
\qquad\quad -\frac{1}{2}\beta x-\frac{1}{2}\gamma x^2-\frac{1}{4}\delta x^3-\frac{1}{12}\epsilon x^4 \text{ \&c.}\\
\qquad\quad +\frac{1}{3}\gamma x+\frac{11}{24}\delta x^2+\frac{7}{24}\epsilon x^3 \text{ \&c.}\\
\qquad\quad -\frac{1}{4}\delta x-\frac{5}{12}\epsilon x^2 \text{ \&c.}\\
\qquad\quad +\frac{1}{5}\epsilon x \text{ \&c.}\\
\qquad\quad \text{\&c.}
\end{array}$$

And, by cor. 5 prop. 1 ſect. 1, the area will be

$$\begin{array}{l}
L x : a+\frac{1}{2}\alpha x+\frac{1}{6}\beta x^2+\frac{1}{24}\gamma x^3+\frac{1}{120}\delta x^4+\frac{1}{720}\epsilon x^5 \text{ \&c.}\\
\qquad\quad -\frac{1}{4}\beta x-\frac{1}{6}\gamma x^2-\frac{1}{16}\delta x^3-\frac{1}{60}\epsilon x^4 \text{ \&c.}\\
\qquad\quad +\frac{1}{6}\gamma x+\frac{11}{72}\delta x^2+\frac{7}{96}\epsilon x^3 \text{ \&c.}\\
\qquad\quad -\frac{1}{8}\delta x-\frac{5}{36}\epsilon x^2 \text{ \&c.}\\
\qquad\quad +\frac{1}{10}\epsilon x \text{ \&c.}\\
\qquad\quad \text{\&c.}
\end{array}$$

Where L repreſents the length of the baſe.

Which area will be had accurately true when the differences are continued till one order of them becomes equal to nothing, for then the ſeries will break off and terminate; that is, an area is quadrable whenever one of the orders of the differences of its equidiſtant ordinates conſiſts of a ſeries of nothings or any other equals, or of a ſeries of arithmeticals; but if an order of the differences never become equal to nothing, the area will be expreſſed by an infinite ſeries.

When the terms or ordinates are taken near to one another, the differences will the ſooner become equal to nothing, or nearly ſo; and if any order of differences, and conſequently its ſucceeding ones, be rejected as inconſiderable, we ſhall have an approximate value of the area, and that the nearer the truth the more of the differences α, β, γ, δ, ϵ, &c. are uſed.

Thus, if there be only one ordinate, or if α, β, γ, &c. be rejected, the area will be aL.

If there be two ordinates, or β, γ, δ, &c. be rejected, x will be $= 1$, and the area will be $\overline{a + \tfrac{1}{2}\alpha} \times L = \frac{a+b}{2} \times L$.

If there be three ordinates, or γ, δ, ϵ, &c. be rejected, x will be $= 2$, and the area will be

$$\overline{a + \alpha + \tfrac{2}{3}\beta - \tfrac{1}{2}\beta} \times L = \overline{a + \alpha + \tfrac{1}{6}\beta} \times L = \frac{a+4b+c}{6} \times L.$$

If there be four ordinates, or δ, ϵ, &c. be rejected, x will be $= 3$, and the area will be

$$\left.\begin{array}{r} \overline{a + \tfrac{3}{2}\alpha + \tfrac{3}{2}\beta + \tfrac{9}{8}\gamma} \\ - \tfrac{3}{4}\beta - \tfrac{3}{2}\gamma \\ + \tfrac{1}{2}\gamma \end{array}\right\} \times L = \overline{a + \tfrac{3}{2}\alpha + \tfrac{3}{4}\beta + \tfrac{1}{8}\gamma} \times L = \frac{a+3b+3c+d}{8} \times L.$$

And thus by putting x equal to every number ſucceſſively, we ſhall have the following table of areas anſwering to the reſpective number of ordinates ſet oppoſite to them, of which every expreſſion is more accurate than the preceding ones, and in which A repreſents the ſum of the firſt and laſt terms, B the ſum of the ſecond and laſt but one, C the ſum of the third and laſt but two, &c. and the laſt of the letters denotes the double of the middle term when the number of terms is odd, alſo L denotes the length of the whole baſe or the diſtance between the firſt and laſt terms.

N°

N° of ordin.	Areas.
1	AL
2	$\frac{A}{2} \times L$
3	$\frac{A + 2B}{6} \times L$
4	$\frac{A + 3B}{8} \times L$
5	$\frac{7A + 32B + 6C}{90} \times L$
6	$\frac{19A + 75B + 25C}{288} \times L$
7	$\frac{41A + 216B + 27C + 136D}{840} \times L$
8	$\frac{751A + 3577B + 1323C + 2989D}{17280} \times L$
9	$\frac{989A + 5888B - 928C + 10496D - 2270E}{28350} \times L$
&c.	&c.

EXAMPLE.

Taking the ſecond example to the laſt propoſition, where are given the five perpendiculars 10, 11, 14, 16, 16, and the diſtance between the firſt and laſt $= 20$; we ſhall have $A = 10 + 16 = 26$, $B = 11 + 16 = 27$, $C = 14 \times 2 = 28$, and $L = 20$.

Hence by the rule for five ordinates $\frac{7A + 32B + 6C}{90} \times L = \frac{182 + 864 + 168}{90} \times 20 = \frac{1214}{9} \times 2 = 269\frac{7}{9} =$ the area, as before nearly.

SCHOLIUM.

When there are many ordinates given, the caſe may be reduced to fewer by adding together the firſt and laſt ordinates, the ſecond and laſt but one, the third and laſt but two, and ſo on, and conſidering the ſums as a new ſet of terms upon a baſe equal to half that of the former. Or there may be added together the two firſt and two laſt terms into one ſum, then the four terms next to theſe, &c. or we may add three at the beginning to three at the end, then the next ſix, &c. always diminiſhing the baſe in proportion to the number of terms that are added into each ſum. And the content will be nearly the ſame in each caſe.

EXAMPLE.

Taking the fourth example to the laſt propoſition, in which are given the eleven ordinates $\frac{10}{10}$, $\frac{10}{11}$, $\frac{10}{12}$, $\frac{10}{13}$, $\frac{10}{14}$, $\frac{10}{15}$, $\frac{10}{16}$, $\frac{10}{17}$, $\frac{10}{18}$, $\frac{10}{19}$, $\frac{10}{20}$, and the diſtance of the firſt and laſt $= 1$; we ſhall have $\frac{10}{10} + \frac{10}{20} = 1{\cdot}5$, $\frac{10}{11} + \frac{10}{19} = 1{\cdot}4354069$, $\frac{10}{12} + \frac{10}{18} = 1{\cdot}3888889$, $\frac{10}{13} + \frac{10}{17} = 1{\cdot}357466$, $\frac{10}{14} + \frac{10}{16} = 1{\cdot}33928 57$, and $\frac{10}{15} \times 2 = 1\frac{1}{3}$.

Hence $A = 1{\cdot}5 + 1\frac{1}{3} = 2\frac{5}{6}$, $B = 1{\cdot}4354069 + 1{\cdot}3392857 = 2{\cdot}7746926$, $C = 1{\cdot}3888889 + 1{\cdot}357466 = 2{\cdot}7463549$, and $L = \frac{1}{2}$.

Then, by the rule for ſix ordinates, we ſhall have $\frac{19A + 75B + 50C}{288} \times L = {\cdot}6931476$, which is much nearer to the truth than the number found by the former rule, the true number being ${\cdot}69314718$, and is the hyperbolic logarithm of 2.

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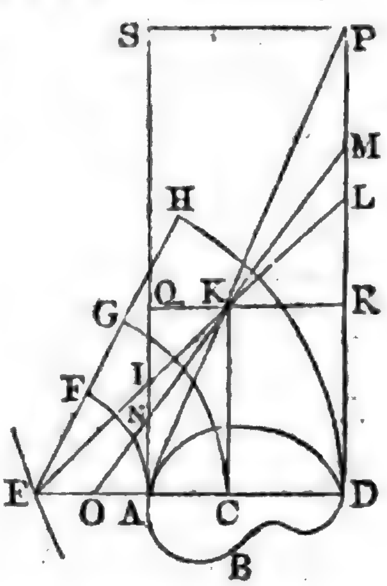
SECTION III.

Of the Relation between the AREAS *and* SOLIDITIES *of Figures and the Centers of Gravity of their generating* PLANES *and* LINES.

PROPOSITION I.

If any line, right or curve, or any plane figure whether it be bounded by right lines or curves, revolve about an axe in the plane of the figure; the ſurface or ſolid generated, will be reſpectively equal to the ſurface or ſolid whoſe baſe is the given line or figure, and height equal to the arc deſcribed by the center of gravity of the ſaid generating line or figure; and conſequently the content will be found by drawing the generating line or figure into the arc deſcribed by its center of gravity.

DEMONSTRATION.



Let *AFHD* be the figure generated by the given line or plane *ABD*; through *C* the center of gravity of which draw *DCAE* perpendicular to the axe of revolution, and meeting with *HGFE* in *E*; and let every point of the baſe be reduced to *AD* by means of perpendiculars to it.

The figure *AFHD* generated is equal to all the *AF*, *CG*, *DH*, &c. But, by ſimilar figures, all the

AF, *CG*, *DH*, &c. are as all the *EA*, *EC*, *ED*, &c. and, by mechanics, the ſum of all the *EA*, *EC*, *ED*, &c. is equal to ſo many times *EC*; therefore the ſum of all the *AF*, *CG*, *DH*, is equal to ſo many times *CG*, or equal to *AD* × *CG*; that is, the figure *AFHD* is equal to *ABD* × *CG* the baſe drawn into the line deſcribed by its center of gravity. *Q.E.D.*

COROLLARY I.

From *E* draw *EIKL* cutting the upright priſmatic figure erected upon the given baſe *ABD*, ſo as that any perpendicular *AI* may be equal to its correſponding arc *AF*. Then will the figure *AILD* be equal to the figure *AFHD*.

For, by ſimilar figures, all the *AF*, *CG*, *DH*, &c. are as all the *AI*, *CK*, *DL*, &c. each to each; and as one of each are equal, therefore they are all equal, each to each; viz. all the *AI*, *CK*, *DL*, &c. equal to all the *AF*, *CG*, *DH*, &c. that is, the figure *AILD* equal to the figure *AFHD*.

COROLLARY II.

Through *K* draw *MKNO*; and the figure *ANMD* will be equal to the figure *AIKLD* or equal to the figure *AFHD*.

For, by the laſt corollary, *ANMD* is equal to the figure deſcribed by the baſe *AD* revolving about *O* till the arc deſcribed by *C* be equal to *CK*; which, by the propoſition, is equal to *AD* × *CK* or *AD* × *CG*.

Co-

COROLLARY III.

Hence all the upright figures *AQKRD*, *AIKLD*, *ANKMD*, *AKPD*, &c. of the ſame baſe, and bounded at the top by lines or planes cutting the upright ſides and paſſing through the extremity *K* of the line *CK* erected upon the center of gravity of the baſe, are equal to one another; and the value of each will be equal to the baſe drawn into the line *CK*.

Hence alſo all figures, deſcribed by the revolution of the ſame line or plane about different centers or axes, will be equal to one another when the arcs deſcribed by the center of gravity are equal. But if thoſe arcs be not equal, the figures generated will be as the arcs. And in general, the figures generated will be to one another as the revolving lines or planes drawn into the arcs deſcribed by their reſpective centers of gravity.

COROLLARY IV.

Moreover, the oppoſite parts *NIK*, *MLK*, of any two of theſe figures are equal to each other.

COROLLARY V.

The figure *ASPD* is to the figure *APD*, as *AS* to *CK*; or, by ſimilar triangles, they will be as *AD* to *AC*.

For *ASPD* is equal to $AD \times AS$, and *APD* is equal to $AD \times CK$.

COROLLARY VI.

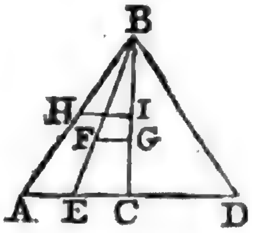
If the line or plane be ſuppoſed to be at an infinite diſtance from the center about which it revolves, the figure generated will be an upright ſurface or priſm, the altitude being the line deſcribed by the center of gravity; ſo that the baſe drawn into the ſaid line will be equal to the baſe drawn into the altitude, as it ought for all upright figures whoſe ſections parallel to the baſe are all equal to each other.

EXAMPLE I.

If a right line or a parallelogram revolve about a line perpendicular to the length, there will be deſcribed a ring either ſuperficial or ſolid; and as the center of gravity of the deſcribing line or parallelogram is in the middle of them, the general rule will become the ſame with rule 3 ſect. 1 part 3, and the rule at prob. 2 ſect. 7 part 3.

When the center of revolution is in the end of the line, the line will deſcribe a circle whoſe radius is the ſaid deſcribing line, and whoſe circumference is double the circumference deſcribed by the center of gravity; conſequently the radius drawn into half the circumference will be the area of the circle.

EXAMPLE II.



If the right-angled triangle ABC revolve about the perpendicular BC and deſcribe the cone ABD.

Draw BE to biſect AC, and take EF equal to one-third of BE; and F will be the center of gravity of the triangle ABC, as is well known. Draw FG parallel to

AC.

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AC.——Then the line deſcribed by F will be the circumference of a circle whoſe radius is FG; and, by the general rule, the cone will be equal to $\triangle ABC \times GF \times 8n$ (putting $n = \cdot 785398$ &c.) $= AC \times \frac{1}{2}CB \times \frac{2}{3}CE \times 8n = AC \times \frac{1}{2}CB \times \frac{1}{3}CA \times 8n = AC^2 \times \frac{1}{3}CB \times 4n = AD^2 \times \frac{1}{3}CB \times n =$ the baſe drawn into one-third of the altitude, as it ought.

Again, from H the middle of AB draw HI parallel to AC; then is H the center of gravity of AB, and conſequently the ſurface deſcribed by AB will be $AB \times$ circumference whoſe radius is $HI = AB \times$ half the circumference whoſe radius is $AC =$ the ſide drawn into half the circumference of the baſe = the ſurface of the cone, as it ought.

EXAMPLE III.

If the ſemi-circle DCA (fig. page 14) revolve about the diameter AD, and deſcribe the ſurface of a ſphere.

If there be taken $DC : CF :: CF : FH = \frac{CF^2}{DC} = \frac{4rr}{c}$, putting r for the radius and c for the whole circumference; H will be center of gravity of the arc DCA, and conſequently $r : c :: (FH)\ \frac{4rr}{c} : 4r =$ the line or circumference deſcribed by H the center of gravity; and, by the general rule, $DCA \times 4r = \frac{1}{2}c \times 4r = 2rc =$ the ſurface of the ſphere = the circumference into the diameter, as it ought.

And for the ſolidity of the ſphere, we ſhall have firſt $\frac{1}{2}c : 2r :: \frac{2}{3}r : \frac{8rr}{3c} =$ the diſtance FH of the center of gravity of the ſemi-circle $DCAD$ from the diameter *AD*, which is two-thirds of the diſtance of the center of gravity of the arc *DCA* from the ſame diameter *DA*, in the former caſe; conſequently the line deſcribed by the center of gravity in this caſe will be two-thirds of *that* in the former; but the deſcribing line in the former caſe is to the deſcribing ſpace in *this* as 1 is to $\frac{1}{2}r$, wherefore $1 : \frac{2}{3} \times \frac{1}{2}r$ :: ſurface of the ſphere : ſolidity $= \frac{1}{3}r \times$ ſurface.

COROLLARY.

The circumference of the circle whoſe radius is the diſtance of the center of gravity of the ſemicircumference of any circle from its center, is equal to four times the radius of that circle.

EXAMPLE IV.

For the ſolidity of the parabolic ſpindle, putting $b =$ the baſe and $a =$ the altitude or axe of the generating parabola, and $n = \cdot785398$, &c. as before.

It is known that $\frac{2}{3}a$ is the diſtance of the center of gravity from the baſe, and conſequently $\frac{16}{5}a =$ the line deſcribed by the center of gravity; but $\frac{2}{3}ab$ is = the revolving area; wherefore $\frac{16}{5}a \times \frac{2}{3}ab = \frac{32}{15}aab$ will be the content, which is $\frac{8}{15}$ of the circumſcribed cylinder.

EXAMPLE V.

For the paraboloid. Making the notation as in the laſt example, $\frac{3}{8}b$ will be the diſtance of the center of gravity of the ſemi-parabola from the axe, conſequently $\frac{3}{8}b \times 8n \times \frac{2}{3}ab = 2abbn =$ the ſolidity = half the circumſcribed cylinder.

PART

PART V.

SECTION I.

Of LAND SURVEYING.

IT is ſuppoſed that land-meaſuring firſt gave riſe to geometry, which hath ſince been gradually riſing to the height at which we at preſent view it. Since the diviſion of common grounds hath become ſo faſhionable in Britain, ſurveying hath been univerſally taught and practiced throughout the nation. The mathematical inſtrument makers have alſo come in for a ſhare of the profits, by introducing a variety of inſtruments, often to little and ſometimes even to bad purpoſe. I ſhall here therefore give a ſhort account of two or three of the moſt uſeful inſtruments.

CHAPTER I.

The deſcription and uſe of the moſt uſeful inſtruments for ſurveying.

I. *Of the Chain.*

Land is meaſured with a chain, called *Gunter's* chain, of 4 poles or 22 yards in length, which is divided into 100 equal parts called links, each of which is in length $\frac{22}{100}$ yards = $\frac{66}{100}$ feet = 7·92 inches; but in turning links into inches or feet, or the contrary, we may, in many caſes, without ſenſible error ſuppoſe each link to be 8 inches or $\frac{2}{3}$ of a foot in length. A ſtatute acre of land is 4840 ſquare yards = 160 ſquare poles or 10 ſquare chains; or ſince 40 ſquare poles or $\frac{1}{4}$ of an acre is called a rood, the table of the diviſions for land meaſure will ſtand thus,

625 ſquare links = 1 pole or perch
40 poles = 1 rood
4 roods = 1 acre
or 100000 ſquare links = 1 acre.

Hence, when any area is found in ſquare links, point off five places towards the right for decimals of an acre, and the figures on the left will be ſo many compleat acres; then value the ſaid five decimal places of an acre, by multiplying them by 4 for roods, and again by 40 for perches.

II. *Of the plain Table.*

The plain table is ſo called either from its being a plain ſurface, or from its reducing the land, in meaſuring, to a plain, whether horizontal or inclined to the horizon.

This inſtrument conſiſts of a plain rectangular board of any convenient ſize, the center of which when uſed is fixed by means of ſcrews to a three-legged ſtand having a ball and ſocket at the top, by means of which when the legs are fixed on the ground the table is inclined in any direction.

To the table belongs,

I. A frame of wood made to fit round its edges, and to take off for the convenience of putting a ſheet of paper upon the table. The one ſide of this frame

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frame is divided into 360 degrees from a center which is in the middle of the table, and the other ſide is divided into 180 degrees to a larger radius from a center towards one ſide of the table; by the means of which diviſions the table is to be uſed as a theodolite, &c.

II. A needle and compaſs ſcrewed into the ſide of the table, to point out the directions and be a check upon the ſights.

III. An index, which is a braſs two-foot ſcale with ſights erected perpendicularly upon the ends. The hairs in theſe ſights and one edge of the index are in the ſame plane, and that edge is called the fiducial edge of the index.

Before you uſe this inſtrument, take a ſheet of paper which will cover it, and wet it to make it expand; then ſpread it flat upon the table, preſſing down the frame upon the edges to ſtretch it and keep it fixed there; and when the paper is become dry, it will by contracting again ſtretch itſelf ſmooth or flat from any cramps or unevenneſs. Upon this paper is to be plotted the plan or form of the thing meaſured.

In uſing this inſtrument, begin at any part of the ground you think the moſt convenient, and make a point upon a convenient part of the paper or table to repreſent that point of the ground; then fix in that point one leg of the compaſſes, and apply thereto the fiducial edge of the index, moving it round till through the ſights you perceive ſome remarkable object, as the corner of a field, &c. and from the ſtation point draw a line with the point of the compaſſes along the fiducial edge of the index; then ſet another object and draw its line; do the ſame by another, and ſo on till as many objects be ſet as may be thought neceſſary. Then meaſure from your ſtation towards ſo many of the objects as may be neceſſary and no more, taking the requiſite offsets to corners or crooks in the hedges, &c. and lay the meaſures down upon their reſpective lines upon the table. Then at any convenient place, meaſured to, fix the table in the ſame poſition, and ſet the objects which appear from thence, &c. as before; and thus continue till your work be finiſhed, meaſuring ſuch lines as are neceſſary, and determining ſo many as you can by interſecting lines of direction drawn from different ſtations.

And in theſe operations obſerve the following particular cautions and directions.

1. Let the lines upon which you make ſtations be directed towards objects as far diſtant as poſſible, and when you have ſet any ſuch object, go round the table and look through the ſights from the other end of the index, to ſee if any other remarkable object lie directly oppoſite; if there be not ſuch an one, endeavour to find another forward object ſuch as ſhall have a remarkable backward oppoſite one, and make uſe of it rather than the other; becauſe the back object will be of uſe in fixing the table in the original poſition either when you have meaſured too near to the forward object, or when it may be hid from your ſight at any neceſſary ſtation by intervening hedges, &c.

2. Let the ſaid lines, upon which the ſtations are taken, be purſued as far as you conveniently

can; for that will be the means of preſerving more accuracy in the work.

3. At each ſtation it will be neceſſary to prove the truth of it; that is, whether it be ſtreight in the line towards the object, and alſo whether the diſtance be rightly meaſured and laid down on the paper.—To know if the table be ſet down ſtreight in the line, lay the index upon the table in any manner, and move the table about till through the ſights you perceive either the fore or back object; then, without moving the table, go round it and look through the ſights by the other end of the index, to ſee if the other object can be perceived; if it be, the table is in the line; if not, it muſt be ſhifted to one ſide, according to your judgment, till through the ſights both objects can be ſeen.—The aforeſaid operation only informs you if the ſtation be ſtreight in the line; but to know if it be in the right part of the line, that is, if the diſtance have been rightly laid down; fix the table in the original poſition, by laying the index along the ſtation line and turning the table about till the fore and back objects appear through the ſights, and then alſo will the needle point at the ſame degree as at firſt; then lay the index over by the ſtation point and any other point on the paper repreſenting an object which can be ſeen from the ſtation; and if the ſaid object appear ſtreight through the ſights, the ſtation may be depended on as right; if not, the diſtance ſhould be examined and corrected till the object can be ſo ſeen. And for this very uſeful purpoſe, it is adviſeable to have ſome high object or two, which can be ſeen from

the greateſt part of the ground, accurately laid down on the paper from the beginning of the ſurvey, to ſerve continually as proof objects.

When from any ſtation the fore and back objects cannot both be ſeen, the agreement of the needle with one of them may be depended on for placing the table ſtreight on the line, and for fixing it in the original poſition.

Of the ſhifting of the paper.

When one paper is full, and you have occaſion for more; draw a line in any manner through the fartheſt point, of the laſt ſtation line, to which the work can be conveniently laid down; then take the ſheet off the table, and fix another on, drawing a line upon it, in the part the moſt convenient for the reſt of the work; then fold or cut the old ſheet by the line drawn on it, apply the edge to the line on the new ſheet, and, as they lie in that poſition, continue the laſt ſtation line upon the new paper, placing thereon the reſt of the meaſure, beginning at where the old ſheet left it. And ſo on from ſheet to ſheet.

After the work is done, when you would faſten all the ſheets together into one piece or rough plan, theſe lines are to be accurately joined together as when the lines were transfered from the old ſheets to the new ones.

But it is to be noted that if the ſaid joining lines, upon the old and new ſheet, have not the ſame inclination to the ſide of the table, the needle will not point to the original degree when the table is rectified; and if the needle be required to reſpect

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ſpect ſtill the ſame degree of the compaſs, the eaſieſt way of drawing the lines in the ſame poſition, is to draw them both parallel to the ſame ſides of the table by means of the equal diviſions marked on the other two ſides.

Of the Theodolite.

The theodolite is a brazen circular ring divided into 360 degrees, having an index with ſights placed upon the center, about which the index is moveable, and a compaſs fixed to the center, to point out courſes and check the ſights; the whole being fixed by the center upon a ſtand of a convenient height for uſe.

In uſing this inſtrument, an exact account, or field-book, of all meaſures and things neceſſary to be remarked in the plan, muſt be kept, from which to make out the plan upon your return home from the ground.

Begin at ſuch part of the ground and meaſure in ſuch directions as you judge moſt convenient, taking angles or directions to objects, and meaſuring ſuch diſtances as appear neceſſary, under the ſame reſtrictions as in the uſe of the plain table. And it is ſafeſt to fix the theodolite in the original poſition at every ſtation by means of fore and back objects, and the compaſs, exactly as in uſing the plain table; regiſtering the number of degrees cut off by the index when directed to each object; and at any ſtation placing the index at the ſame degree as when the direction towards that ſtation was taken from the laſt preceding one, to fix the theodolite there in the original poſition, after the ſame manner as the plain table is fixed in the original poſition by laying the index along the line of the laſt direction.

The beſt method of laying down the aforeſaid lines of direction, is to deſcribe a pretty large circle, quarter it, and lay upon it the ſeveral numbers of degrees cut off by the index in each direction; then, by means of a parallel ruler, draw from ſtation to ſtation lines parallel to lines drawn from the center to the reſpective points in the circumference.

Of the Croſs.

The croſs conſiſts of two pair of ſights ſet at right angles to each other, upon a ſtaff having a ſharp point to ſtick in the ground.

The croſs is very uſeful to meaſure ſmall and crooked pieces. The method is to meaſure a baſe or chief line, generally in the longeſt direction of the piece from corner to corner, taking the places of perpendiculars upon this line from the ſeveral corners and bends in the boundary of the piece, with the croſs, by fixing it by trials upon ſuch parts of the line as that through one pair of the ſights both ends of the line may appear, and through the other pair you can perceive the correſponding bends or corners; and then meaſuring the lengths of the ſaid perpendiculars.

SCHOLIUM.

Beſides the fore-mentioned inſtruments, which are moſt uſed, there are ſome others, as the circumferentor much like to the theodolite in ſhape and

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and uſe, and the ſemi-circle for the taking of angles, &c. But of all the inſtruments for meaſuring, the plain table is certainly the moſt excellent; not only becauſe it may be uſed as a theodolite or ſemi-circle, according as you put uppermoſt the ſide of the frame having the 360 degrees or that which hath the ſemi-circle or 180; but becauſe that it is in its own proper uſe by much the eaſieſt, ſafeſt, and moſt accurate for the purpoſe; for by planning every part immediately upon the ſpot as ſoon as meaſured, there is not only ſaved a great deal of writing in the field book, but becauſe that every thing can be planned more eaſily and accurately while it is in view than afterwards from a field book, in which many little things muſt be either neglected or miſtaken; and beſides, the opportunities which the plain table afford of correcting our work or proving if it be right at every ſtation, are ſuch advantages as can never be balanced by any other method. But although the plain table be the moſt generally uſeful inſtrument, it is not *always* ſo; there being many caſes in which ſometimes one inſtrument is the propereſt and ſometimes another; nor is that ſurveyor maſter of his buſineſs who cannot in any caſe diſtinguiſh which is the fitteſt inſtrument or method, and uſe it accordingly: nay ſometimes no inſtrument at all, but barely the chain itſelf, is the beſt method, particularly in regular open fields lying together; and even when you are uſing the plain table it is often of advantage to meaſure ſuch with the chain only and from thoſe meaſures lay them down upon the table; and ſome eminent ſurveyors have been ſo ſtruck with the eaſe and ſimplicity of this method, that they have neglected and exploded the uſe of every inſtrument whatever: and indeed ſuch practice is not ſo abſurd as ſome other; for although it be often attended with much trouble in meaſuring, and even ſometimes be impracticable, as in ſurveying a town, or among buſhes, &c. and attended with trouble alſo in the planning on account of ſome of the lines meaſuring to more than they ought; yet when ſuch lines are corrected and the parts of the plan made to fit, the true ſurface will then be repreſented by the plan: but of all the abſurd or falſe methods *that* by the uſe of what is called the new-improved theodolite is of the moſt dangerous conſequence; for it is profeſſedly adapted to the diminiſhing of all land by reducing it to an *horizontal* plain, even ſuch as is wholly upon a *regular* declivity; by that means deſtroying ſometimes a third or fourth of the true ſurface; it is the ſurface that ſurveyors have to deal with, without any regard to its inclination, except indeed where a pointed hill, &c. ariſes out of a plain, for ſuch all the methods would level down to the general ſurface; nor is it to any purpoſe to ſay that the horizontal plain will bear as many perpendiculars or piles of graſs, &c. as the ſlope will do, for that the ſurveyor has no buſineſs with, let him give the true ſurface, and leave the owner to rate and let his land according to its quality. In ſhort, let every gentleman, who would not have his real land reduced below its juſt quantity, beware of thoſe ſurveyors who uſe this new-improved inſtrument.

CHAPTER II.

Of ſome particular methods of meaſuring, illuſtrated by examples.

Beſides the general methods of meaſuring, mentioned in the preceding deſcription of inſtruments, may be obſerved the following particular caſes of meaſuring and of computing the contents.

Many pieces of land may be very well meaſured by meaſuring any baſe line, either within or without them, together with the perpendiculars let fall upon it from every corner of them. For they are by thoſe means divided into ſeveral triangles and trapezoids all whoſe parallel ſides are perpendicular to the baſe line; and the ſum of theſe triangles and trapeziums will be equal to the figure propoſed if the baſe line fall within it; if not, the ſum of the parts which are without being taken from the ſum of the whole which are both within and without, will leave the figure propoſed.

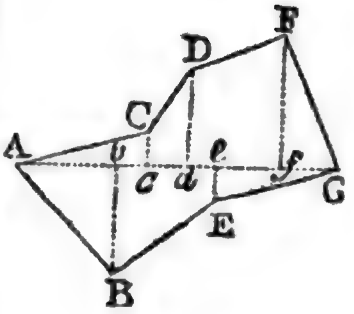
In pieces that are not very large, it will be ſufficiently exact to find the points, in the baſe line, where the ſeveral perpendiculars will fall, by means of the *Croſs*, and from thence meaſuring to the corners for the lengths of the perpendiculars.—— And it will be moſt convenient to draw the line ſo as that all the perpendiculars may fall within the figure.

EXAMPLE I.

Having, in the field *ACDFGEBA*, meaſured the baſe line *AG* and perpendiculars as are expreſſed in the following table or field-book, in links: Required the content, and plot of the field.

Field-Book.

Baſe line	Perpendiculars
$Ab = 315$	$bB = 350$ South
$Ac = 440$	$cC = 70$ North
$Ad = 585$	$dD = 320$ N.
$Ae = 610$	$eE = 50$ S.
$Af = 990$	$fF = 470$ N.
$AG = 1020$	0



The calculation.

$Ab \times bB$ — $= 315 \times 350 = 110250 = 2AbB$.
$Ac \times cC$ — $= 440 \times 70 = 30800 = 2AcC$.
$cd \times \overline{cC + Dd} = 145 \times 390 = 56550 = 2cCDd$.
$be \times \overline{bB + Ee} = 295 \times 400 = 118000 = 2bBEe$.
$df \times \overline{dD + fF} = 405 \times 790 = 319950 = 2dDFf$.
$Ff \times FG$ — $= 470 \times 30 = 14100 = 2fFG$.
$Ge \times eE$ — $= 50 \times 410 = \underline{20500} = 2GEe$.

the ſum = $\underline{670150}$ = double the whole figure; whoſe half = 335075 ſquare links = 3·35075 acres = 3A. 1R. 16·12 Perches = the area required.

To plot the figure above, or any other meaſured in the ſame manner.

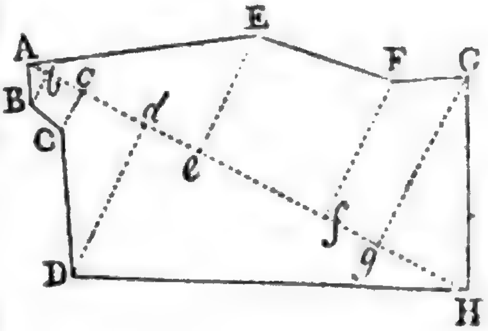
Draw an indeterminate line *AG* for the baſe line, and transfer upon it, from a ſcale of equal parts, the ſeveral diſtances *Ab*, *Ac*, *Ad*, &c. equal to the given meaſures 315, 440, 585, &c. Then, at the points *b*, *c*, *d*, &c. erect the perpendiculars *bB*, *cC*, *Dd*, &c. equal to the given lengths 350, 70, 320, &c. on the upper or under ſide of the baſe line according as they are north or ſouth. Laſtly, draw the lines *AC*, *CD*, *DF*, &c. and it will be done.

EXAMPLE II.

Required the area and plot of the figure from the following field-book.

Note, That a cypher in the place of a perpendicular, denotes that there the baſe line touches an angular point. So here the firſt and laſt perpendiculars, being cyphers, ſhew that the baſe line begins and ends at an angle.

Field-Book.	
Baſe line	Perpendiculars
0	0
19	42 S.
80	62 S.
184	235 S.
307	184 N.
470	220 N.
556	270 N.
697	0



Calculation.

$$\begin{aligned}
&Ab \times bB \ — = 19 \times 42 = 0798 = 2ABb.\\
&bc \times \overline{cC + Bb} = 61 \times 104 = 6344 = 2BCcb.\\
&cd \times \overline{dD + Cc} = 104 \times 297 = 30888 = 2CDdc.\\
&Dd \times dH \ — = 235 \times 513 = 120555 = 2DdH.\\
&Ae \times eE \ — = 307 \times 184 = 56488 = 2EeA.\\
&ef \times \overline{fF + Ee} = 163 \times 404 = 65852 = 2FEef.\\
&fg \times \overline{gG + Ff} = 86 \times 490 = 42140 = 2GFfg.\\
&Gg \times gH \ — = 270 \times 141 = \underline{38070} = 2HGg.
\end{aligned}$$

the ſum = 361135, the half of which = 180567·5 ſquare links = 1·805675 acres = 1 A. 3 R. 8·908 P.

But in large pieces or tracts of land it is better to reduce the meaſures to a baſe line drawn in a direction exactly eaſt-and-weſt, and touching the ſouthmoſt angle of the figure.——And this is Mr. *Thomas Burgh*'s "method to determine the areas of right-lined figures univerſally," and is performed thus.

Begin at any corner, and, going round the figure, meaſure the length of each ſide and the angle formed by each and the meridian or north-and-ſouth line; then the figure being plotted, draw the baſe line, in an eaſt-and-weſt direction, through the ſouthmoſt corner, and let fall perpendiculars, or ſouth lines, from every ſtation or angular point upon the baſe line. Then multiply the ſum of the perpendiculars from every two adjacent ſtations into the diſtance, in the baſe line, intercepted between them: Collect into one ſum all the products belonging to every two ſtations of which the latter ſhall be to the eaſt of the former, and collect into another ſum all the other products; and half the difference of the ſums will be the area of the figure.*

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* For, beginning at the weſt at *A* (ſee the figure to the following example) and proceeding round by *B*, *C*, &c. it is plain that $\overline{Aa+bB}\times ab =$ double the trapezoid $AabB$, $\overline{Bb+cC}\times bc =$ double $BbcC$, $\overline{Cc+dD}\times cd =$ double $CcdD$, $\overline{Ee+fF}\times ef =$ double $EefF$; and that the ſum of theſe products will be the double of the figure $aABCDEFfa$ together with the double of $EedD$, becauſe that $EedD$ is included in both $CcdD$ and $EefF$.

Again $\overline{Dd+eE}\times de =$ twice $DdeE$, $\overline{Ff+gG}\times fg =$ twice $FfgG$, $Gg\times gH =$ twice GgH, $Hi\times iI =$ twice HiI, $\overline{Ii+aA}\times ia =$ twice $IiaA$; and the ſum of theſe products make up the doubles of the two figures $AaHIA$, $HfFGH$, together with the double of $EedD$ alſo.

Conſequently the latter ſum taken from the former will leave the area of the figure $ABCDEFGHIA$ required.

Note. The perpendiculars upon the baſe line from the ſeveral ſtations Mr. Burgh calls their abſolute nothings; and the diſtances between the perpendiculars, the particular eaſtings and weſtings of the ſtations, viz. eaſting when the ſtation is to the eaſt of the preceding ſtation, and weſting when it is to weſt of it. Then his rule is to multiply every particular eaſting between two ſtations into the half ſum of the abſolute northings, collecting the products into one ſum; and every particular weſting between two adjacent ſtations into the half ſum of their abſolute nothings, collecting theſe products alſo: then the difference of the ſums is the area.

EXAMPLE III.

Suppoſe that there is to be meaſured the following figure by this method.

Beginning at the weſtmoſt ſtation A, and going round towards the north, ſuppoſe the lengths of the lines and the angles formed by them and the meridians to be taken thus: $AB = 1550$ links, and its direction N. $35\frac{1}{2}^\circ$ E. that is, $35\frac{1}{2}^\circ$ from the north towards the eaſt; I enter theſe, then, in the 2d and 3d columns, of the following table, oppoſite to the 1ſt ſtation A. Then proceeding to the 2d ſtation B, I find the direction of BC to be N. E. $72\frac{3}{4}^\circ$, and its length = 1870 links, which, therefore, I enter in the table upon a line with the ſecond ſtation B. In the ſame manner I find all the other ſides and their directions as expreſſed in order in the 2d and 3d columns of the table; the 1ſt column containing only the number and mark of each ſtation.

Then by conſtructing the figure, which is very eaſy to do, and meaſuring, by the ſcale, the lengths and diſtances of the ſeveral perpendiculars, and multiplying, &c. according to rule, we ſhall have the area pretty near.

But to have the area exact, the ſeveral perpendiculars and their diſtances muſt be calculated by trigonometry, or, for more expedition, by a traverſe table.——Thus, in the triangle ABz, the diſtance AB being 1550, and the courſe or angle at $B = 35\frac{1}{2}°$, I find, in the traverſe table, the eaſting $Az = 900$, and northing $zB = 1262$; theſe I enter, in my table, in the columns of northing and eaſting, marked N and E, upon a line with the firſt ſtation *A*. Then, in the triangle BCy, to the diſtance 1870 and courſe $72\frac{3}{4}°$, I find the eaſting $By = 1786$, and the northing $yC = 555$, which I place, in the columns *E* and *N*, oppoſite the 2d ſtation. In like manner, in the triangle CxD, I find the ſouthing $Cx = 616$, and eaſting $xD = 1765$, which I place accordingly in the columns of ſouthing and eaſting. And in the triangle DEw, the ſouthing $Dw = 749$, and weſting $wE = 994$, which I put in the columns S and W. And ſo on till, by going through all the triangles, I return to the point *A* where I began, placing the particular northings or ſouthings, eaſtings or weſtings in the table.——And it is evident that, being returned to the firſt ſtation *A*, the ſum of all the northings muſt be equal to the ſum of the ſouthings, and the ſum of the eaſtings equal to that of the weſtings; and, accordingly, by ſumming them up in the table, I find that they are ſo, the ſum of each of the former being

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being 3982, and that of each of the latter 6844, which is a very good proof of this part of the work.

Then, for the abſolute northings, it is evident that the ſum of all the northings 3982 muſt be equal to *cC* the abſolute northing of the northmoſt ſtation *C*; and this I place on the 8th column of the table, marked Abſol. N. being the column of abſolute northings, oppoſite the ſtation *C*. Then *Cx*, the particular ſouthing of *C*, taken from its abſolute northing, gives the abſolute northing *dD* of the next ſtation *D*: again, the particular ſouthing of *D* taken from its abſolute northing, gives the abſolute northing of the next ſtation *E*: and ſo on, to or from the abſolute northing of every ſtation, adding its particular northing or ſubtracting its particular ſouthing, for the abſolute northing of the next ſucceeding ſtation. And theſe are all entered, in the column of abſolute northings, oppoſite their ſtations.

In the 9th column, tituled *Sums*, are placed the ſums of every two abſolute northings oppoſite the upper of each of ſuch abſolute northings.——Each of theſe ſums being multiplied into the particular eaſting or weſting upon the ſame line with it, produces the double areas of the ſeveral trapezoids, and are placed in the two laſt columns of the table; viz. the products ariſing from the multiplication of the eaſtings being written in the former of them, and the others in the latter. Then theſe products being ſummed, half their difference is 145A. 1R. 32P. the area ſought.

Stations	Angles	Sides	N	S	E	W	Abfol. Ns.	Sums	E. Prod.	W. Prod.
1. *A*	NE $35\frac{1}{2}$°	1550	1262		900		2165 = *aA*	5592	5032800	
2. *B*	NE $72\frac{3}{4}$	1870	555		1786		3427 = *bB*	7409	13232474	
3. *C*	SE $70\frac{3}{4}$	1870		616	1765		3982 = *cC*	7348	12969220	
4. *D*	SW 53	1245		749		994	3366 = *dD*	5983		5947108
5. *E*	SE $83\frac{1}{4}$	2410		283	2393		2617 = *eE*	4951	11847743	
6. *F*	SW $31\frac{1}{4}$	1520		1299		789	2334 = *fF*	3369		2658141
7. *G*	SW $62\frac{3}{4}$	2260		1035		2008	1035 = *gG*	1035		2078280
8. *H*	NW $73\frac{1}{2}$	2730	775			2617	0 = *hH*	775		2026635
9. *I*	NW $17\frac{5}{12}$	1456	1399			436	775 = *iI*	2940		1281840
			3982	3982	6844	6844			43082237	13992004

13992004

2)29090233

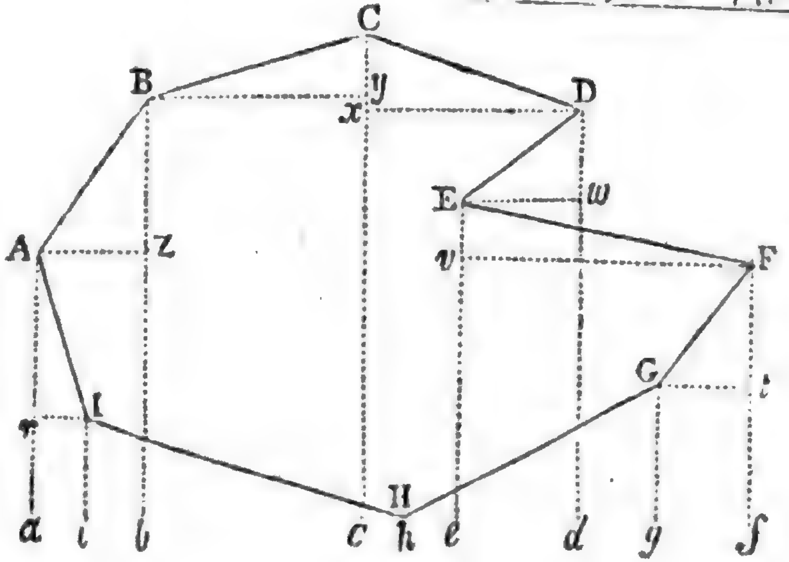
145·451165

4

1·804660

40

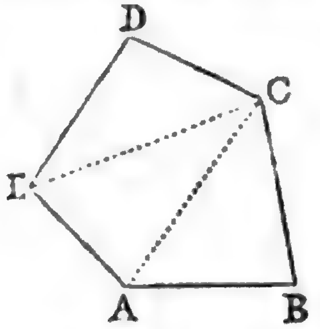
32·1864



Some other examples, promiſcuouſly ſet down.

EXAMPLE IV.

In a pentangular field, beginning with the ſouth ſide, and meaſuring round towards the eaſt firſt, the 1ſt or ſouth ſide was = 2735 links, the 2d = 3115, the 3d = 2370, the 4th = 2925, and the 5th = 2220; alſo the diagonal from the 1ſt angle to the 3d was 3800, and that from the 3d to the 5th was 4010; Required the figure and area.



Having conſtructed the figure, I firſt take the diſtance of the angle *B* from the diagonal *AC* between the compaſſes, and, by applying them to the ſcale, find it to be 2210, which is the length of the perpendicular from *B* upon *AC*. Wherefore $3800 \times 2210 = 8398000$ double the triangle *ABC*.

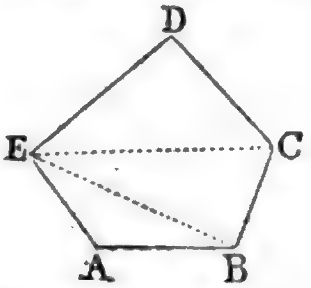
Then in the ſame manner I find the diſtances of *A* and *D* from the diagonal *CE* to be 2065 and 1710. Wherefore $4010 \times \overline{2065 + 1710} = 4010 \times 3775 = 15137750$ = double the trapezium *ACDE*.

Whence the ſum of the two products is 23535750, the half of which is 11767875 ſquare links, equal 117 A. 2 R. 28·6 P. the area ſought.

Note. As this figure conſiſts of three triangles all whoſe ſides are given, by calculating their areas by rule 2 of problem 2, their ſum will be that of the whole figure accurately, and you will then find how much the area above aſſigned differs from the truth, which, I believe, will be but about the 30th part of an acre.

EXAMPLE V.

In going round a five-ſided field *ABCDE*, the ſides and angles were thus: The ſide $AB = 1940$ links, and the angle B 110° 30′; the ſide $BC = 1555$, and the angle C 117° 45′; the ſide $CD = 2125$, and the angle D 91° 20′; and the ſide $DE = 2741$: Required the figure and the content.



Having conſtructed the figure and drawn the two diagonals *EB*, *EC*, which meaſure 3180 and 3500, I meaſure the diſtance between *A* and the diagonal *EB*, and find it = 780, and the diſtances of *B* and *D* from the diagonal *EC* are 1430 and 1670.

Then $3180 \times 780 = 2480400 =$ double the $\triangle ABC$,

And $3500 \times \overline{1430 + 1670} = 3500 \times 3100 = 10850000$ = double the trapezium *EBCD*.

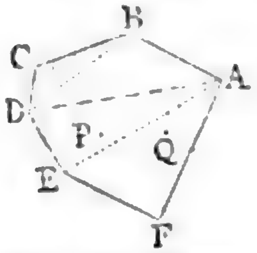
Wherefore $\frac{10850000 + 2480400}{2} = 6665200$ ſquare links = 66 A. 2 R. 24·32 P. the area ſought.

EXAMPLE VI.

In a field I took two ſtations P and Q, at the diſtance of 10 chains from each other:

From P I took the angles		And from Q the angles	
	$QPA = 21^\circ\ 20'$		$PQD = 10^\circ\ 40'$
	$APB = 49\ \ 10$		$DQC = 18\ \ 30$
	$BPC = 57\ \ 12$		$CQB = 42\ \ 00$
	$CPD = 29\ \ 40$		$BQA = 67\ \ 05$
	$DPE = 64\ \ 25$		$AQF = 137\ \ 00$
	$EPF = 79\ \ 16$		$FQE = 62\ \ 52$

Required the figure and area.



Having conſtructed the figure, by the line AD, I divide it into two trapeziums $ABCD$, $ADEF$: In the former of which the diagonal BD meaſures 1860, and the perpendiculars 1420 and 440; and in the latter, the diagonal AE = 2940, and the perpendiculars 1050 and 1350.

$$\text{Whence } 1860\times\overline{1420+440}=1860\times1860=3459600$$
$$\text{and } 2940\times\overline{1050+1350}=2940\times2400=\underline{7056000}$$
$$2)\underline{10515600}$$

Acres 52·578
4
Roods 2·312
40
Perches 12·48

A. R. P.
Anſ. 52 2 12

If from a ſtation within a field, the diſtances to the ſeveral angles be meaſured, as alſo the angles formed by theſe diſtances.

Then multiply the product of every two adjacent diſtances by the natural ſine of the angle between them, and half the ſum of theſe products will be the *exact* area.

EXAMPLE VII.

From a ſtation within a field of five ſides I meaſured the diſtances to the ſeveral corners, beginning with that on the weſt, and meaſuring round towards the north, viz. 1ſt diſtance 7345 links, the 2d = 5980, the 3d = 6495, the 4th = 6015, and the 5th = 7050; alſo the angles formed by theſe diſtances in the ſame order were $71\frac{1}{2}$, $55\frac{3}{4}$, $49\frac{1}{4}$, and $81\frac{1}{2}$ degrees. What is the area.

(Sine of $71\frac{1}{2}°$) ·9483237 × 7345 × 5980 = 41653316·25

(— – $55\frac{3}{4}$) ·8265897 × 5980 × 6495 = 32104826·57

(— – $49\frac{1}{4}$) ·7575650 × 6495 × 6015 = 29596113·97

(— – $81\frac{1}{2}$) ·9890159 × 6015 × 7050 = 41939961·01

(— – 78) ·9781476 × 7050 × 7345 = 50650683·55

2)195944901·35

Acres 979·72450675

4

Roods 2·89802700

40

Perches 35·92108

Note. The laſt angle (78°) is found by ſubtracting the ſum of all the other angles from 360°; for all the angles formed by lines about a point make up four right angles or 360°.

Ex-

EXAMPLE VIII.

From a place *O*, near the middle of a field *ABCDEF*, from which I could ſee all the angles, I meaſured the diſtances to the ſeveral angles, and obſerved the quantities of the angles formed at *O* by thoſe diſtances as below.

Diſtances	Angles
$OA = 4315$ links	$AOB = 60° \; 30'$
$OB = 2982$	$BOC = 47 \; 40$
$OC = 3561$	$COD = 49 \; 50$
$OD = 5010$	$DOE = 57 \; 10$
$OE = 4618$	$EOF = 64 \; 15$
$OF = 3606$	$FOA = 80 \; 35$

What is the area.

(Sine of 60° 30′) ·8703557×4315×2982=11199154·17

(— — 47 40) ·7392394×2982×3561= 7849910·77

(— — 49 50) ·7641714×3561×5010=13633284·15

(— — 57 10) ·8402513×5010×4618=19440205·26

(— — 64 15) ·9006982×4618×3606=14998884·03

(— — 80 35) ·9865246×3606×4315=15350214·66

2)82471653·04

Acres 412·3582652

4

Rood 1·4330608

40

Perches 17·322432

EXAMPLE IX.

Having made choice of a ſtation within a piece of land, I meaſured from thence the ſeveral bearings and diſtances of the corners as below: Required the the area.

	Bearings	Diſtances
1ſt	N.E.	3540
2d	N.½E.	4785
3d	N.W.b.W.	3915
4th	S.W.b.S.	4125
5th	S.S.E. 7° E.	2030
6th	E.b.S. 3½E.	2945

The ∠ between the 1ſt & 2d diſt. is 3½ points or 39° 22′½
2d & 3d — 5½ — or 61 52½
3d & 4th — 8 — or 90 00
4th & 5th — 5 pts. 7° or 63 15
5th & 6th — 4 pts. 7¾ or 52 45
6th & 1ſt — 4 pts. 7¾ or 52 45
Sum 32 pts. or 360 00

Then

(ſine of 39° 22′½) ·6343932×3540×4785=10745923·32
(— — 61 52½) ·8819212×4785×3915=16521272·60
(— — 90 00) 1·0000000×3915×4125=16149375·00
(— — 63 15) ·8929789×4125×2030= 7477582·14
(— — 52 45) ·7960020×2030×2945 } 13057338·26
(— — 52 45) ·7960020×2945×3540 }

2)63951491·32
Acres 319·7574566
4
Roods 3·0298264
40
Perches 1·193056

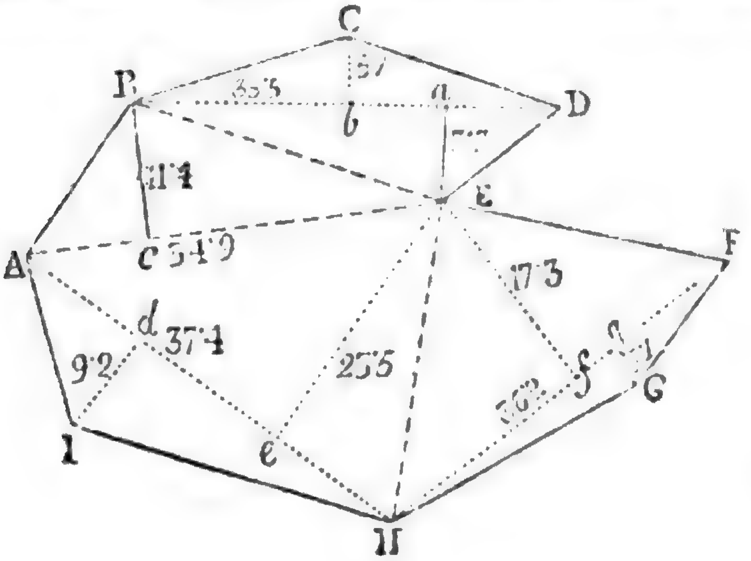
CHAPTER III.

Of the general method of caſting up the contents of ſurveys.

After the ſurvey is finiſhed and the whole put together into one rough plan, the general method of caſting up the contents, is to divide the whole into trapeziums and triangles, by drawing lines from angle to angle, and meaſuring the neceſſary parts of theſe figures, by taking their lengths between the compaſſes and applying them to the ſcale from which the whole figure was laid down; and then computing their contents by their proper rules laid down in former parts. By theſe means the work is very expeditiouſly done, and ſufficiently correct; for ſuch dimenſions are taken as afford the moſt eaſy method of calculation; and, among a number of parts thus taken and applied to a ſcale, it is likely that ſome will be taken too little and ſome too great, ſo that they will, upon the whole, in all probability, nearly balance one another.

EXAMPLE I.

Suppoſe that, after having planned the figure in the 3d example, I would calculate it in this manner.



Firſt, then, from the angle *E* I draw lines to the angles *B*, *A*, *H*, dividing the figure into three trapeziums *EDCB*, *EAIH*, *EHGF*, and a triangle *EBA*. Then I draw the diagonals *DB*, *AH*, *HF*, and let fall the perpendiculars *Ea*, *Cb*, *Bc*, *Id*, *Ee*, *Ef*, *Gg*.

Then in the trapezium *EDCB*, the diagonal meaſures 3550 links and the perpendiculars 570 and 770. Wherefore $3550 \times \overline{570 + 770} = 3550 \times 1340 = 47570$ = double the trapezium *EDCB*.

And, in the triangle *EBA*, *AE* meaſures 3490 and the perpendicular 1140, their product is 39786 equal double the triangle.

Alſo, in the trapezium *EAIH*, the diagonal meaſures 3740, and the perpendiculars 920 and 2350. Wherefore $3740 \times \overline{920 + 2350} = 3740 \times 3270 = 122298$ equal double this trapezium.

Laſtly, in the trapezium *EHGF*, the diagonal meaſures 3620, and the perpendiculars 1730 and 510. Wherefore $3620 \times \overline{1730 + 510} = 3620 \times 2240 = 81088$ equal double that trapezium.

47570
39786
122298
81088

2)290742 double the whole area in links

Acres 145·371
4

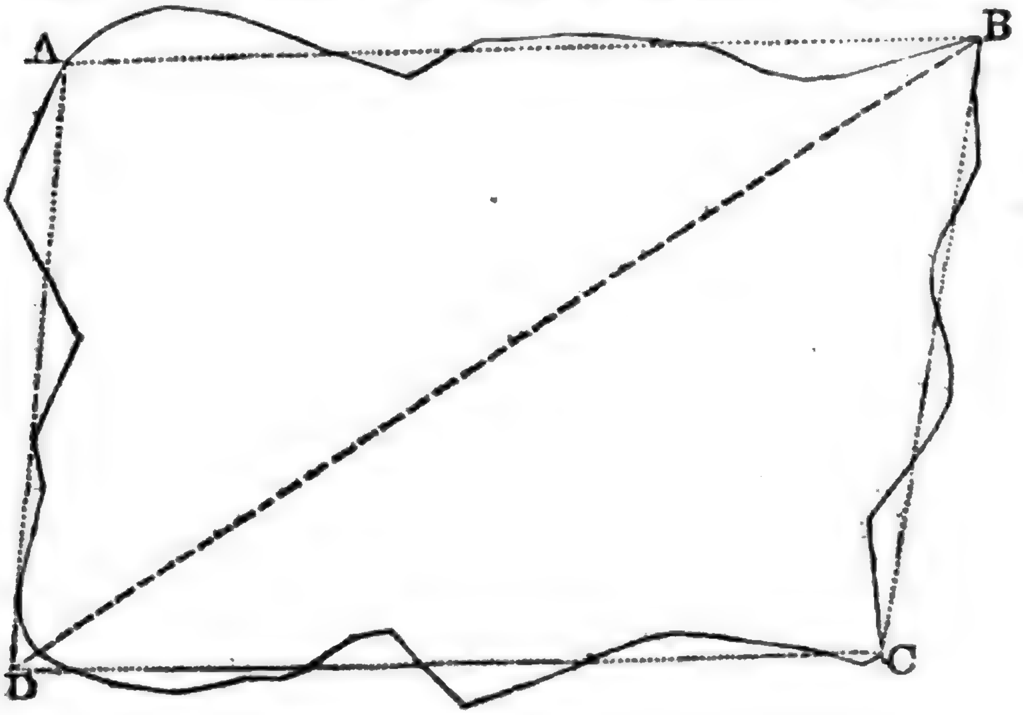
Roods 1·484
40

Perches 19·36

Hence the area by this method is 145A. 1R. 19P. and is leſs than the former by 13 perches only, which is leſs than $\frac{1}{12}$ part of an acre; a very inconſiderable quantity in ſo large a figure.

But the chief ſecret of caſting up conſiſts in finding the contents of pieces bounded by curved or very irregular lines, or in reducing thoſe lines to ſtreight ones, and is eaſieſt and very accurately performed thus: Stretch a horſe hair over the crooked ſides ſo as that the ſmall parts cut off the figure by it may be equal to thoſe which are taken in, which equality of the parts you will preſently be able to judge of by a little practice; then mark the ends of the hair with two points, and with a pencil draw a line through thoſe two points: Do the ſame by the next adjoining ſide, &c. till the whole be reduced to a ſtreight-ſided figure; then find its content as above.

Thus if it were required to find the content of the following figure bounded by the irregular black line, from a ſcale of 4 chains to the inch.



I draw the four lines *AB*, *BC*, *CD*, *DA* cutting off equal quantities on each ſide of them, by which means the irregular figure is reduced to the right-lined trapezium *ABCD* as nearly equal to it as can be judged. Then, having drawn the diagonal *DB*, by meaſuring I find the diſtance of *A* from *DB* to be 790, and the diſtance of *C* from $DB = 760$; whoſe ſum 1550 drawn into $DB = 1916$ produces 2969800, the half of which or 1484900 ſquare links = 14A. 3R. 15·84P. = the content required.

NOTE that the eaſieſt method of uſing the hair is to ſtring a ſmall bow of wire, cane, or whale-bone with it; for, *this* keeping it always ſtretched, it can be uſed with one hand, while the other is at liberty to make the marks.

CHAPTER IV.

Of Planning.

The ornamenting of plans, as well as the chief part of the ſubject of ſurveying, muſt be got by much practice together with proper inſtructions and examples from a living preceptor; and all that is proper to be done here, is to give a few general remarks, together with the general method of taking off the fair plan from the rough one.

The neateſt and eaſieſt method of taking the clean plan off from the rough one, is thus: Rub the back of the rough plan over with black lead powder; and lay the ſaid black part upon the clean paper upon which the plan is to be copied, and in the proper poſition; then with the blunt point of ſome hard ſubſtance, as braſs, &c. trace over the lines, &c. of the whole plan, preſſing the tracer ſo much as that the black lead under the lines may be tranſferred to the clean paper; after which take off the rough plan, and trace over the leaden marks with common or with india ink, &c.——Or, inſtead of blacking the rough plan, you may keep conſtantly a blacked paper to lay between the plans.

Another method of copying plans is by means of ſquares; which is performed by dividing both ends and ſides of the plan to be copied into any convenient number of equal parts, and connecting the correſponding points of diviſion with lines, which will divide the plan into a number of little ſquares; divide the paper upon which the plan is to be copied into the ſame number of ſquares, each equal to the former when the plan is to be copied of the ſame ſize, but greater or leſs than the others in the proportion in which the plan is to be increaſed or diminiſhed, when of a different ſize; then copy into the clean ſquares the parts contained in the correſponding ſquares of the old plan, and you will have the plan copied either of the ſame ſize, or greater or leſs in any proportion.

A third method is by the inſtrument called a pentagraph, which alſo copies the plan in any ſize required.

When the lines, &c. are copied upon the clean paper or vellum, the next buſineſs is to write ſuch names, remarks, or explanations as may be judged neceſſary; laying down the ſcale for taking the lengths of any parts, a flower-de-luce to point out the direction, and the proper title ornamented with a compartment; and illuſtrating or colouring every part in ſuch manner as ſhall ſeem moſt natural, ſuch as ſhading rivers or brooks with crooked lines, drawing the repreſentations of trees, buſhes, hills, woods, hedges, houſes, gates, roads, &c. in their proper places; running a ſingle dotted line for a foot path, and a double one for a carriage road; and either repreſenting the baſes or the elevations of buildings, &c.—But to do all theſe things neatly, you ought to be properly inſtructed in the art of drawing.

CHAPTER V.

Of the Diviſion of Lands.

In the diviſion of commons, after the whole is ſurveyed and caſt up, and the proper quantities to be allowed for roads, &c. deducted, divide the neat quantity remaining among the ſeveral proprietors, by the following problem, in proportion to the real value of their eſtates, and you will thereby obtain their proportional quantities of the land. But as this diviſion ſuppoſes the land, which is to be divided, to be all of an equal goodneſs, you muſt obſerve that if the part in which any one's ſhare is to be marked off, be better or worſe than the general mean quality of the land, then you muſt diminiſh or augment the quantity of his ſhare in the ſame proportion as near as you can judge.

PROBLEM I.

It is required to divide any given quantity of ground into any given number of parts, and in proportion as any given numbers.

RULE.

Divide the given piece after the rule of fellowſhip, by dividing the whole content by the ſum of the numbers expreſſing the proportions of the ſeveral ſhares, and multiplying the quotient ſeverally by the ſaid proportional numbers for the reſpective ſhares required.

EXAMPLE.

It is required to divide 300 acres of land among *A*, *B*, *C*, *D*, *E*, *F*, *G*, and *H*, whoſe claims upon it are reſpectively in proportion as the numbers 1, 2, 3, 5, 8, 10, 15, 20.

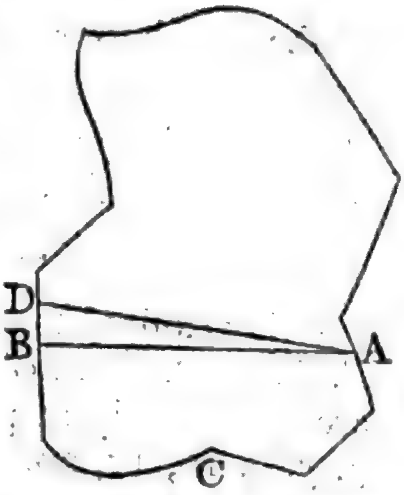
The ſum of theſe proportional numbers is 64, by which dividing 300, the quotient is 4·6875, which being multiplied by each of the numbers 1, 2, 3, 5, &c. we obtain for the ſeveral ſhares as below:

A =	4·6875 =	4 A.	2 R.	30 P.
B =	9·3750 =	9	1	20
C =	14·0625 =	14	0	10
D =	23·4375 =	23	1	30
E =	37·5000 =	37	2	00
F =	46·8750 =	46	3	20
G =	70·3125 =	70	1	10
H =	93·7500 =	93	3	00
Sum =	300·0000 =	300	0	00

PROBLEM II.

To cut off from a plan a given number of acres, &c. by a line drawn from any point in the ſide of it.

Let *A* be the given point in the annexed plan, from which a line is to be drawn cutting off 5 A. 2 R. 14 P.



Draw AB cutting off the part ABC as near as can be judged equal to the quantity propoſed, and let the true quantity of ABC, when calculated, be only 4A. 3R. 20P. which is leſs than 5A. 2R. 14P. the true quantity, by 0A. 2R. 34P. or 71250 ſquare links; then meaſure AB, which ſuppoſe = 1234 links, and by 617 the half of which divide 71250, the quantity to be added, and the quotient 115 links will be the altitude of the triangle to be added and whoſe baſe is AB; wherefore if upon the center B with the radius 115 an arc be deſcribed, and AD be drawn to touch that arc, AD will be the line cutting off the required quantity $ADCA$.

NOTE that if the firſt piece had been too big, then D muſt have been ſet below B.

In this manner the ſeveral ſhares of commons to be divided may be laid down upon the plan, and transferred from thence to the ground itſelf.

Alſo for the greater eaſe and perfection in this buſineſs, I ſhall ſubjoin the following problems.

PROBLEM III.

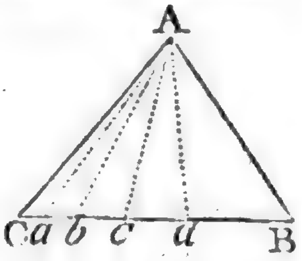
From an angle in a given triangle to draw lines to the oppoſite ſide, dividing the triangle into any number of parts, which ſhall be in any aſſigned proportion to each other.

RULE.

Divide the baſe into the ſame number of parts and in the ſame proportion, by problem 1; then from the ſeveral points of diviſion draw lines to the propoſed angle, and they will divide the triangle as required.*

EXAMPLE.

Let the triangle ABC of 20 acres be divided into five parts which ſhall be in proportion to the numbers 1, 2, 3, 5, 9, the lines of diviſion being to be drawn from A to BD whoſe length is 1600 links.

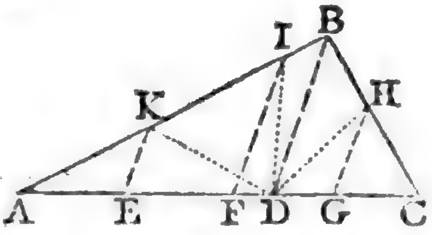


Here $1+2+3+5+9=20$, and $1600 \div 20 = 80$, which being multiplied by each of the proportional numbers, we have 80, 160, 240, 400, and 720; wherefore I make $Ca = 80$, $ab = 160$, $bc = 240$, $cd = 400$, and $dB = 720$; then by drawing the lines Aa, Ab, Ac, Ad, the triangle is divided as required.

PROBLEM IV.

From any point in one ſide of a given triangle, to draw lines to the other two ſides, dividing the triangle into any number of parts which ſhall be in any aſſigned ratio.

RULE.



From the given point D draw DB to the angle oppoſite to the ſide AC in which the point is taken; then divide the ſame ſide AC

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* DEMONSTRATION.

For the ſeveral parts are triangles of the ſame altitude, and which therefore are as their baſes, which baſes are taken in the aſſigned proportion.

AC into ſo many parts *AE*, *EF*, *FG*, *GC*, and in the ſame proportion with the required parts of the triangle, like as was done in the laſt problem; and from the points of diviſion draw lines *EK*, *FI*, *GH*, parallel to the line *BD*, and meeting with the other ſides of the triangle in *K*, *I*, *H*; laſtly, draw *KD*, *ID*, *HD*, ſo ſhall *ADK*, *KDI*, *IDHB*, *HDC* be the parts required.*

The example to this will be done exactly as the laſt.

I omit here the diviſion of other figures into either two or more parts, becauſe the method of doing it is ſo much limited as to render it of no uſe in practice; I omit alſo their reduction to triangles, as that is done only for the ſake of their diviſion; and ſhall add here only one problem more, ſhewing how readily to increaſe or diminiſh a figure by changing it from one ſcale to another; but when large or complex ſurveys are to be changed to a different ſcale, a pentagraph ought to be uſed for the purpoſe, or elſe the method of ſquares as deſcribed in chapter 4.

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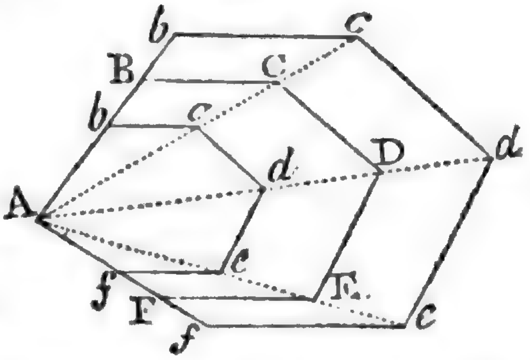
* DEMONSTRATION.

The triangles *ADK*, *KDI*, *IDB*, being of the ſame height, are as their baſes *AK*, *KI*, *IB*, which, by means of the parallels *EK*, *FI*, *DB*, are as *AE*, *EF*, *FD*; in like manner, the triangles *GDH*, *HDB* are to each other as *CG*, *GD*: but the two triangles *IDB*, *BDH*, having the ſame baſe *BD*, are to each other as the diſtances of *I* and *H* from *BD*, or as *FD* to *DG*; conſequently the parts *DAK*, *DKI*, *DIBH*, *DHC* are to each other as *AE*, *EF*, *FG*, *GC*.

PROBLEM V.

To change a figure from one ſcale to another.

RULE.



From one angle *A* draw lines *AC*, *AD*, *AE*, &c. to all the other angles of the figure given; then augment or diminiſh one ſide *AB* till *Ab* be to *AB* in the given proportion of the ſcales; and, by means of a parallel ruler, draw *bc* parallel to *BC* and meeting *AC* in *c*, and in the ſame manner *cd* parallel to *CD*, *de* parallel to *DE*, *ef* parallel to *EF*, ſo ſhall *AbcdefA* be the figure required.

SECTION II.

Of Caſk Gauging.

THE meaning of the word *Gauging* is reſtricted to the meaſuring of caſks and other things falling under the cognizance of the officers of the exciſe; and hath received its name from a guage or rod uſed by the practioners of the art.

The buſineſs being performed, or the calculations made, generally, by means of the inſtruments, called the gauging or diagonal rod, and the ſliding rule or gauging rule, it will be neceſſary to treat of theſe inſtruments, which I ſhall do as below.

CHAPTER I.

The Deſcription and Uſe of the Sliding Rule.

This is a ſquare rule having conſequently four ſides or faces, three of which are furniſhed with ſliding pieces running in grooves. The lines upon them are moſtly logarithmic ones, or diſtances which are proportional to the logarithms of the numbers placed at the ends of them; which kind of lines were placed upon rules, by Mr *Edmund Gunter*, for expeditiouſly performing arithmetical operations; in which buſineſs he uſed a pair of compaſſes for taking the ſeveral logarithmic diſtances: but inſtead of the compaſſes, ſliding pieces were added, by Mr *Thomas Everard*, as being more convenient and certain in practice.

Upon the firſt face are three lines, marked *A*, *B*, and *MD*; *B* being upon the ſlider, and is a kind of double line, being marked at both the edges of the ſlider, for the convenience of application to both the lines *A*, and *MD*. Theſe three are ſuch logarithmic lines as are deſcribed above, and are all of the ſame radius or diſtance from 1 to 10, containing each twice the length of the radius; *A* and *B* are placed and numbered exactly alike, the numbering beginning at 1, by which means the middle diviſion will be 10 times and the laſt 100 times the value aſſigned to the firſt diviſion, which, both here and in all logarithmic lines, may have any value put upon it at pleaſure, either 1, 10, 100, 1000, &c. or ·0, ·01, ·001, ·0001, &c. But the number 1 upon the line *MD* is ſet oppoſite to the diviſion 215 or more exactly

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actly 215042 on the other lines, and then numbered in the order retrograde to that of the lines *A* and *B*; by which means it happens that not only both the diviſions, on *MD*, marked 1 are oppoſite to 215042 on the lines *A* and *B*, but alſo that all the three diviſions, on *A* and *B*, marked 1 are oppoſite to the ſame diviſion 215042 on *MD*. The reaſon of this diſpoſition of the line *MD* is that 2150·42 cubic inches are contained in a malt buſhel; this line in conjunction with *B* ſerving to gauge malt, while the lines *A*, *B*, ſerve for operations in multiplication and diviſion. Moreover upon the lines *A*, *B*, are ſeveral other marks and letters: viz. Upon the line *A* are the letters *MB*, for malt buſhel, at the aforeſaid number 2150·42; and at 282, the inches in an ale gallon, are a mark and the letter *A*, denoting ale: and upon the line *B* are *si*, for ſquare inſcribed, and a mark at ·707 &c. the ſide of a ſquare inſcribed in a circle whoſe diameter is 1; and *se*, for ſquare equal, and a mark at ·886 &c. the ſide of a ſquare equal to the ſame circle; alſo *C*, for circumference, and a mark at 3·14159 &c. the circumference of the ſame circle; and moreover *W*, for wine, and a mark at 231, the inches in a wine gallon.

The ſecond face, being that oppoſite to the firſt, is marked *D*, *C*, *D*, *E*, at one end, and Root, Square, Root, Cube, at the other, the lines *C* and *E* containing reſpectively the ſquares and the cubes of the diviſions ſet oppoſite to them on the lines *D*; the two middle lines *C*, *D*, being on a ſlider. Theſe, alſo, are logarithmic lines, but the radius of *D* is double to that of *A*, *B*, and *C*, and triple to that of *E*. And whatever the firſt 1 on *D* denotes, the firſt on *C* is the ſquare of it, and the firſt on *E* the cube of it; ſo if *D* begin with 1, *C* will begin with 1 and *E* with 1; but if *D* begin with 10, *C* will begin with 100 and *E* with 1000. Upon the line *D* are ſeveral points marked, viz. at 17·15, the wine gauge point for circles, is placed *WG* for wine gauge; at 18·95 is marked *AG* for ale gauge; at 46·37, the malt gauge point for ſquare meaſure, is ſet *MS* for malt ſquare; and at 52·32 is put *MR*, in like manner, for malt round or circular meaſure; alſo at 6·23, the neat tallow gauge point, are placed the letters *TP* for tallow pounds. Moreover upon the line *C* is marked *oc* at ·07957, and *od* at ·7854, the former being the area of a circle whoſe circumference is 1, and the latter that of a circle whoſe diameter is 1.

Upon the third face are three lines; one, upon a ſlider, marked *N*, which is a logarithmic line of the ſame radius or ſize with *A*, *B*, *C*, and *MD*; and two others, marked *S. S.* and *S. L.* denoting ſegment ſtanding and ſegment lying, this face being for ullaging ſtanding and lying caſks.

And upon the fourth or oppoſite face are a ſcale of inches with three other ſcales, marked ſpheroid or 1ſt variety, 2d variety, 3d variety; the line for the fourth or conic variety being upon the inſide of the ſlider in the third face. The uſe of theſe lines is to find the mean diameters of caſks.

Beſides the above lines, are two others upon the inſides of the two firſt ſliders, being continued from the one ſlider upon the other; the one is a ſcale of inches from 12½ to 36, and the other a ſcale of ale gallons from the correſponding numbers ·435 to 3·61

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ale gallons; which form a table to ſhew, in ale gallons, the contents of all cylinders whoſe diameters are from 12½ to 36 inches, the common height of them being 1 inch.

As the ſliding rule is for performing, very expeditiouſly, any operations of multiplication, diviſion, and extraction of roots, required by any rule propoſed in words &c. ſo the manner of making theſe operations will appear in the following problems.

PROBLEM I.

To find the product of two given numbers, by the ſliding rule.

RULE. To either of the given numbers on *A* ſet 1 on *B*, then againſt the other number on *B* is the product on *A*.

EXAMPLE I. Required the product of 12 and 25.

By placing 1 on *B* under 12 on *A*, above 25 on *B* ſtands 300 on *A*, which is the product required.

NOTE. When the 1 on *B* hath been ſet to the one factor on *A*, if it happen that the other factor on *B* fall beyond the diviſion on either *A* or *B*, divide it by 10, or 100, &c. till the quotient found on *B* fall under ſome diviſion on the line *A*, and multiply this ſaid diviſion by the ſame 10, or 100, &c. for the product required.

EXAMPLE II. So when 250 is to be multiplied by 56: Having ſet 1 on *B* to 250 on *A*, although 56 be found on *B*, it is beyond the end of *A*; therefore dividing it by 10, I find that oppoſite to the quotient 5·6 on *B* is the diviſion 1400 on *A*, which being multiplied by 10, we obtain 14000 for the product required.

Example III. But if 250 were to be multiplied by 1120: Having ſet 1 to 250 as before, 1120 is beyond the end of *B*, but being divided by 100, oppoſite to the quotient 11·2 on *B* I find 2800 on *A*, which being multiplied by 100 we have 280000 for the product required.

Problem II.

To find the quotient of two numbers.

Rule. Set 1 on *B* to the diviſor on *A*, then againſt the dividend on *A* is the quotient on *B*.

Example I. To divide 300 by 25. Having ſet 1 on *B* to 25 on *A*, oppoſite to 300 on *A* I find 12 on *B*, the quotient required.

Note. When the dividend falls beyond the end of the line *A*, let it be divided by 10, 100, or ſome other power of 10 till it fall within the line, and uſe the quotient inſtead of it, multiplying the reſult by the ſame power of 10 as before.

Example II. So if 14000 muſt be divided by 56. Having ſet 1 to 56, the dividend cannot be found on *A* till it be divided by 100, the quotient being 140, oppoſite to which I find 2·5 on *B*, which being multiplied by 100 we obtain 250 for the quotient required.

Problem III.

To work the rule-of-three on the ſliding rule; or having three numbers given, to find a fourth which ſhall be to the third as the ſecond is to the firſt.

Rule. Set the firſt term on *B* to either the ſecond or third on *A*, then againſt the remaining term on *B* ſtands the fourth term required on *A*.

Example. If 8 yards of cloth coſt 24 ſhillings, what will 96 yards coſt at the ſame rate.

Having ſet 8 on *B* to 24 on *A*, oppoſite to 96 on *B*, I find, on *A*, 288 ſhillings or 14l. 8s. which is the anſwer.

PROBLEM IV.

To extract the ſquare root by the ſliding rule.

Rule. The firſt 1 on *C* ſtanding againſt the firſt 1 on *D* (on the ſtock), oppoſite to the given number on *C* is its root on *D*.

Example. To find the ſide of a ſquare, which ſhall be equal to a triangle, or circle, &c. whoſe area is 225; or, to extract the root of 225.—Here oppoſite to 225 on *C* ſtands 15 on *D*, which is the anſwer required.

PROBLEM V.

To extract the cube root by the ſliding rule.

Rule. The line *D* upon the ſlide being ſet ſtreight with *E*; find the given number on *E*, and oppoſite to it will be its cube root on *D*.

Example. To find the ſide of a cube equal to any other ſolid whoſe content is 3375; or to find the cube root of 3375.—Here oppoſite to 3375 on *E* ſtands 15 on *D*, which is the anſwer required.

Note. It is evident that the ſame lines, as are uſed in theſe two laſt problems, will ſerve to find the ſquare or the cube of any given number, by taking the given number on the contrary lines.

PROBLEM VI.

To find a mean proportional between two given numbers.

RULE. Set one of the given numbers on *C* to the like or ſame number on *D*, then againſt the other given number on *C* is the number required on *D*.

EXAMPLE. To find the ſide of a ſquare whoſe area ſhall be equal to that of a parallelogram whoſe length is 9 and its breadth 4 feet; or, to find a mean proportional between 4 and 9.

Having ſet 4 on *C* to 4 on *D*, againſt 9 on *C* ſtands 6 on *D*, which is the number ſought.

PROBLEM VII.

To find a number which ſhall be to a given number in a given duplicate proportion; or, having given three numbers, to find a fourth, which ſhall be to the third as the ſquare of the ſecond is to the ſquare of the firſt.

RULE. Set the third number on *C* to the firſt on *D*, then againſt the ſecond on *D* will be found, on *C*, the fourth required.

EXAMPLE. If the area of a parallelogram or any other figure be 120; it is required to find the area of a ſimilar figure, their like dimenſions or ſides being as 2 to 3.

Similar figures being as the ſquares of their like dimenſions, by ſetting 120 on *C* to 2 on *D*, againſt 3 on *D* ſtands 270 on *C*, for the number ſought.

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PROBLEM VIII.

To find a number which ſhall be to a given number in a given ſubduplicate proportion; or, having given three numbers, to find a fourth, which ſhall be to the third as the root of the ſecond is to the root of the firſt.

RULE. Set the firſt number on *C* to the third on *D*, then againſt the ſecond on *C* will be found the fourth on *D*.

EXAMPLE. The ſide of a regular figure is 2, and its area 120; it is required to find the ſide of a ſimilar figure whoſe area is 270.

The roots of the areas of ſimilar figures being as their ſides, we muſt find a number which ſhall be to 2 as the root of 270 is to the root of 120. Wherefore, having ſet 120 on *C* to 2 on *D*, againſt 270 on *C* will be found 3 on *D*, which is the number ſought.

PROBLEM IX.

To find a number in a given triplicate proportion to a number given; or, having three numbers given, to find a fourth, which ſhall be to the third as the cube of the ſecond is to the cube of the firſt.

RULE. Set the firſt number on the ſlide *D* to the third number on *E*, then againſt the ſecond on *D* is the fourth required on *E*.

EXAMPLE. If a caſk, whoſe length is 40 inches, contain 100 gallons, what will be the content of a ſimilar caſk whoſe length is 36 inches.

Similar ſolids being as the cubes of their like ſides, the content required muſt be to 100 gallons as 40^3 is to 36^3. Wherefore ſetting 40 on *D* to 100 on *E*, againſt 36 on *D* will be found 72·9 gallons on *E*, which is the content required.

PROBLEM X.

To find a number in a given ſubtriplicate proportion to a given number; or, having three numbers given, to find a fourth, which ſhall be to the third as the cube root of the ſecond is to the cube root of the firſt.

RULE. Set the third number on *D* to the firſt on *E*, then againſt the ſecond on *E* will ſtand the fourth on *D*.

EXAMPLE. What is the length of a caſk whoſe content is 72·9 gallons, ſuppoſing the length of a ſimilar caſk to be 40 inches and its content 100 gallons.

Since the dimenſions of ſimilar ſolids are as the cube roots of their contents, we muſt find a number which ſhall be to 40 as the cube root of 72·9 is to the cube root of 100. Wherefore, having ſet 40 on *D* to 100 on *E*, againſt 72·9 on *E* will be found 36 on *D*, which is the length required.

PROBLEM XI.

The length and breadth of a parallelogram being given, to find its area in malt buſhels by the line MD.

RULE. Set either of the given dimenſions on *B* to the other on *MD*, then againſt 1 on *A* is the required area on *B*.

EXAMPLE. How many malt buſhels can be contained on every inch of the depth of a ciſtern whoſe length is 180 and breadth 72 inches.

By ſetting 72 on *B* to 180 on *MD*, againſt 1 on *A* will appear nearly 6 buſhels on *B*, which is the quantity required.

PROBLEM XII.

To find, by the line MD, *the malt buſhels which may be contained in a couch, floor, or ciſtern, whoſe length, breadth, and depth are given.*

RULE. Set one of the dimenſions on *B* to another on *MD*, then againſt the third on *A* will appear the content on *B*.

EXAMPLE. Required the number of buſhels in the ciſtern whoſe length is 230, breadth 58·2, and depth 5·4 inches.

Having ſet 230 on *B* to 5·4 on *MD*, againſt 58·2 on *A* I find 33·6 buſhels on *B*, which is the content nearly.

The uſe of the other parts or marks on the rule will appear in the examples farther on.

CHAPTER II.

Of the Gauging or Diagonal Rod.

The diagonal rod is a ſquare rule having four ſides or faces, being generally four feet long, and folding together by means of joints.

It takes its name from its uſe in meaſuring the diagonals of caſks, and computing the contents from the ſaid diagonal only; where it may be noted, that by the diagonal of a caſk is meant the line from the bung to the interſection of the head with the ſtave oppoſite to it, and is generally the longeſt line that can be drawn from the bung to any part within the caſk.

And, accordingly, upon one face of the rule is a ſcale of inches for taking the meaſure of the diagonal; to which is adapted the areas, in ale gallons, of circles to the correſponding diameters, like to the lines on the under ſides of the three ſlides in the ſliding rule.

Upon the oppoſite face are two ſcales of ale and wine gallons, expreſſing the contents of caſks having the correſponding diagonals; and theſe are the lines which chiefly conſtitute the difference between this inſtrument and the ſliding rule; for all the other lines upon it are the ſame with thoſe on that inſtrument, and are to be uſed in the ſame manner.

EXAMPLE.

Let it be required to find the content of a caſk whoſe diagonal meaſures 34·4 inches, which agrees with the caſk in the following chapter, whoſe head and bung diameters are 32 and 24, and length 40 inches; for if to the ſquare of 20, half the length, be added the ſquare of 28, half the ſum of the diameters, the ſquare root of the ſum will be 34·4 nearly.

Now to this diagonal 34·4, correſponds, upon the rule, the content 90$\frac{1}{4}$ ale or 111 wine gallons; which differs from all the contents, in the next chapter, obtained by conſidering the caſk as belonging

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longing to each of the four propoſed varieties. So that it is evident there are many caſes in which the diagonal line cannot exhibit the true contents; however in moſt caſes I do not doubt but that it may be fully as accurate as gueſſing at the form or variety of the caſk, and then computing the content by the rule proper to that variety.

CHAPTER III.

Of Caſks conſidered as divided into ſeveral Varieties.

According to the cuſtom of moſt writers on this ſubject, I ſhall diſtinguiſh caſks into four forms or varieties, viz.

1. The middle fruſtum of a ſpheroid,
2. The middle fruſtum of a parabolic ſpindle,
3. The two equal fruſtums of a paraboloid,
4. The two equal fruſtums of a cone.

The middle fruſtums of circular, elliptic, and hyperbolic ſpindles, I omit here on account of the difficulty of their rules, which renders them unfit for the purpoſe of practical gauging. And indeed ſome of the above four forms are of very little real uſe; for very few, if any, caſks are to be met with which will hold ſo much as the firſt form, or ſo little as the third or fourth; ſo that the ſecond is the moſt generally, if not the only uſeful one, of the four varieties.

Note. 282 cubic inches make one ale gallon,
231 — — — wine ——,
2150·42 — — a malt buſhel.

It is alſo to be noted that the dimenſions are ſuppoſed to be inches, in the following rules.

PROBLEM I.

To find the content of a caſk of the firſt or ſpheroidal variety.

RULE.*

To the ſquare of the head diameter add twice the ſquare of the bung diameter, and multiply the ſum by the length of the caſk; then if the product be $\begin{array}{l}\text{mult.}\\ \text{by}\end{array}\left\{\begin{array}{l}\cdot00092837\\ \cdot00113333\end{array}\right\}$ or $\begin{array}{l}\text{div.}\\ \text{by}\end{array}\left\{\begin{array}{r}1077\cdot157\\ 882\cdot355\end{array}\right\}$ for $\left\{\begin{array}{c}\text{ale}\\ \text{wine}\end{array}\right\}$, the product or quotient will be the meaſure in gallons.

BY THE SLIDING RULE.

Set the length of the caſk on *C* to $\left\{\begin{array}{l}32\cdot82\\ 29\cdot7\end{array}\right\}$ for $\left\{\begin{array}{c}\text{ale}\\ \text{wine}\end{array}\right\}$ on *D*, and on *D* find the bung and head diameters, noting the numbers oppoſite to them on *C*; then if the latter of theſe two numbers be added to the double of the former, the ſum will be the meaſure in gallons.

EXAMPLE.

Required the content of a ſpheroidal caſk whoſe bung and head diameters are 32 and 24, and length 40 inches.

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* For, by prob. 12 ſect. 3 part 3, the content in inches is $\frac{2B^2+H^2}{3}$ $\times Ln$, which being divided by 282 and 231, becomes $\frac{2B^2+H^2}{1077\cdot157}\times L$ or $\overline{2B^2+H^2}\times\cdot00092837L$ in the one caſe, and $\frac{2B^2+H^2}{882\cdot355}\times L$ or $\overline{2B^2+H^2}\times\cdot00113333$ in the other; B being the bung and H the head diameter, and L the length of the caſk.——And in working by the ſliding rule, it need only be remarked that 32·82 and 29·7 are the roots of 1077·157 and 882·355.

By the Pen. Here $\overline{2 \times 32^2 + 24^2} \times 40 \times \left\{\begin{matrix}\cdot 00092837 \\ \cdot 00113333\end{matrix}\right\} = \left\{\begin{matrix}97\cdot 4417 \text{ ale} \\ 118\cdot 9543 \text{ wine}\end{matrix}\right\}$ gallons = the content required.

By the Sliding Rule. Having ſet 40 on *C* to 32·82 on *D*, againſt 32 and 24 on *D* ſtand 38 and 21·3, as near as can be judged, on *C*; then $2 \times 38 + 21\cdot 3 = 76 + 21\cdot 3 = 97\cdot 3$ ale gallons.

And having ſet 40 on *C* to 29·7 on *D*, againſt 32 and 24 on *D* ſtand 46·5 and 26·1 on *C*; then $2 \times 46\cdot 5 + 26\cdot 1 = 93 + 26\cdot 1 = 119\cdot 1$ wine gallons.

PROBLEM II.

To find the content of a caſk of the ſecond or parabolic ſpindle form.

RULE.*

To the ſquare of the head diameter add twice that of the bung diameter, and from the ſum take two-fifths of the ſquare of the difference of the ſaid diameters; then multiply the remainder by the length, and the product multiplied or divided by the ſame numbers as in the rule to the laſt problem will give the content.

BY THE SLIDING RULE.

As in the laſt problem, ſet the length on *C* to 32·82 or 29·7 on *D*, and on *D* find both the bung and head diameters, and alſo their difference, taking out the three numbers oppoſite to them on *C*; then if to twice the firſt be added the ſecond, and two-fifths of the third be taken from the ſum, the remainder will be the content.

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* For, by prob. 18 ſect. 4 part 3, the content in inches is $\frac{8B^2+4BH+3H^2}{15} \times Ln = \frac{Ln}{3} \times \overline{\frac{10B^2+4BH+5H^2}{5} - \frac{2B^2+2H^2}{5}} = \frac{Ln}{3} \times \overline{2B^2+H^2-\tfrac{2}{5}\times\overline{B-H}|^2}$, and $\frac{n}{3}$ will give the ſame numbers as in the laſt problem.

EXAMPLE.

Required the content of a caſk of the ſecond variety, whoſe bung and head diameters are 32 and 24, and length 40 inches.

By the Pen. Here $\overline{2\times32^2+24^2-\tfrac{2}{5}\times8^2}\times40\times\left\{\begin{matrix}{\cdot}00092837\\{\cdot}00113333\end{matrix}\right\}=\left\{\begin{matrix}96{\cdot}491\text{ ale}\\117{\cdot}793\text{ wine}\end{matrix}\right\}$ gallons = the content required.

By the Sliding Rule. Having ſet 40 on *C* to 32·82 on *D*, againſt 32, 24, and 8 on *D* ſtand 38, 21·3, and 2·4; then $2\times38+21{\cdot}3-\tfrac{2}{5}\times2{\cdot}4=76+21{\cdot}3-0{\cdot}9=96{\cdot}4$ ale gallons.

And, having ſet 40 on *C* to 29·7 on *D*, againſt 32, 24, and 8 on *D* ſtand 46·5, 26·1, and 2·9; then $2\times46{\cdot}5+26{\cdot}1-\tfrac{2}{5}+2{\cdot}9=93+26{\cdot}1-1{\cdot}2=117{\cdot}9$ wine gallons.

PROBLEM III.

To find the content of a caſk of the third or paraboidal variety.

RULE.

To the ſquare of the bung diameter add that of the head diameter, and multiply the ſum by the length; then if the ſum be $\begin{matrix}\text{mult.}\\\text{by}\end{matrix}\left\{\begin{matrix}{\cdot}00139255\\{\cdot}0017\end{matrix}\right\}\begin{matrix}\text{or div.}\\\text{by}\end{matrix}\left\{\begin{matrix}718{\cdot}105\\588{\cdot}233\end{matrix}\right\}\text{ for }\left\{\begin{matrix}\text{ale}\\\text{wine}\end{matrix}\right\}$, the product or quotient will be the content required.

By

BY THE SLIDING RULE.

Set the length on *C* to $\left\{\begin{matrix}26\cdot8\\24\cdot25\end{matrix}\right\}$ on *D* $\left\{\begin{matrix}\text{for ale}\\\text{for wine}\end{matrix}\right\}$; then find the bung and head diameters on *D*, noting the two oppoſite numbers on *C*, whoſe ſum will be the content required.*

EXAMPLE.

Required the content of a caſk of the third variety, whoſe bung and head diameters are 32 and 24, and length 40 inches.

By the Pen. Here $\overline{32^2+24^2}\times40\times\left\{\begin{matrix}\cdot00139255\\\cdot0017\end{matrix}\right\}$ $=\left\{\begin{matrix}89\cdot1232\text{ ale}\\108\cdot8\text{ wine}\end{matrix}\right\}$ gallons = the content required.

By the Sliding Rule. Having ſet 40 on *C* to 26·8 on *D*, againſt 32 and 24 on *D* ſtand 57·3 and 32; whoſe ſum is 89·3 ale gallons.

And having ſet 40 on *C* to 24·25 on *D*, againſt 32 and 24 on *D* ſtand 69·8 and 39·1, whoſe ſum is 108·9 wine gallons.

PROBLEM IV.

To find the content of a cask of the fourth or conical variety.

RULE.

To three times the ſquare of the ſum of the diameters add the ſquare of the difference of the diameters;

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* For, by prob. 12 ſect. 4 part 3, the content in inches is $\frac{B^2+H}{2}\times Ln$; and $\frac{n}{2\times282}=\frac{1}{718\cdot105}=\cdot00139255$, and $\frac{n}{2\times231}=\frac{1}{588\cdot233}=\cdot0017$.

Alſo the numbers 26·8 and 24·25 are the roots of the numbers 718·105 and 588·233.

ameters; then if the ſum be $\begin{matrix}\text{mult.}\\ \text{by}\end{matrix}\left\{\begin{matrix}\cdot00023209\\ \cdot00028333\end{matrix}\right\}$ or $\begin{matrix}\text{div.}\\ \text{by}\end{matrix}\left\{\begin{matrix}4308\cdot628\\ 3529\cdot42\end{matrix}\right\}$ for $\left\{\begin{matrix}\text{ale}\\ \text{wine}\end{matrix}\right\}$ the product or quotient will be the content required.

BY THE SLIDING RULE.

Set the length on *C* to $\left\{\begin{matrix}65\cdot64\\ 59\cdot41\end{matrix}\right\}$ for $\left\{\begin{matrix}\text{ale}\\ \text{wine}\end{matrix}\right\}$ on *D*, and on *D* find the ſum and the difference of the diameters, noting their oppoſite numbers on *C*; then if the ſecond be added to three times the firſt, the ſum will be the content.*

EXAMPLE.

Required the content of a caſk of the fourth variety, whoſe bung and head diameters are 32 and 24, and length 40 inches.

By the Pen. Here $\overline{3\times56^2+8^2}\times40\times\left\{\begin{matrix}\cdot00023209\\ \cdot00028333\end{matrix}\right\}$ $=\left\{\begin{matrix}87\cdot9342\text{ ale}\\ 107\cdot348\text{ wine}\end{matrix}\right\}$ gallons = the content required.

By the Sliding Rule. Having ſet 40 on *C* to 65·64 on *D*, againſt 56 and 8 on *D* ſtand 29·1 and 0·6; then $3\times29\cdot1+0\cdot6=87\cdot3+0\cdot6=87\cdot9$ ale gallons.

And, having ſet 40 on *C* to 59·41 on *D*, againſt 56 and 8 on *D* ſtand 35·6 and 0·7; then $3\times35\cdot6+0\cdot7=107\cdot5$ wine gallons.

CHAP-

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* For, by prob. 8 ſect. 1 part 3, the content in inches is $\frac{B^2+BH+H^2}{3}\times Ln=\frac{Ln}{12}\times\overline{3\times\overline{B+H}|^2+\overline{B-H}|^2}$; where $\frac{n}{12\times282}=\frac{1}{4308\cdot628}=\cdot00023209$, and $\frac{n}{12\times231}=\frac{1}{3529\cdot42}=\cdot00028333$. Alſo 65·64 and 59·41 are the roots of the numbers 4308·628 and 3529·42.

CHAPTER IV.

Of gauging Casks by their mean diameters.

PROBLEM I.

To find the mean diameter of a cask of any of the four varieties, having given the bung and head diameters.

RULE.*

Divide the head by the bung diameter, and find the quotient in the firſt column of the following table, marked Qu. then if the bung diameter be multiplied by the number upon the ſame line with it, and in the column anſwering to the proper variety, the product will be the true mean diameter required, or the diameter of a cylinder of the ſame content with the caſk propoſed.

Qu.

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* *The Inveſtigation of the numbers of this table.*

By the 3d part it appears that if the bung diameter be repreſented by 1, and the head diameter by b, then the content for the 1ſt, 2d, 3d, and 4th varieties will be reſpectively ln drawn into

$$\left.\begin{array}{c}\dfrac{2+bb}{3},\\[1ex] \dfrac{8+4b+3bb}{15},\\[1ex] \dfrac{1+bb}{2},\\[1ex] \dfrac{1+b+bb}{3},\end{array}\right\}$$

but the content muſt alſo be equal to ln drawn into the ſquare of the mean diameter; and conſequently the ſaid mean diameter, or multiplier in the table, will be reſpectively

$$\left\{\begin{array}{c}\sqrt{\dfrac{2+bb}{3}},\\[1ex] \sqrt{\dfrac{8+4b+3bb}{15}},\\[1ex] \sqrt{\dfrac{1+bb}{2}},\\[1ex] \sqrt{\dfrac{1+b+bb}{3}},\end{array}\right.$$

Then by writing, in theſe forms, the ſeveral values of b, viz. ·50 ·51, ·52, &c. there will reſult the correſponding numbers of the table.——A farther diſſertation upon theſe mean diameters may be ſeen in *Moſs's* gauging.

Qu.	1 Var.	2 Var.	3 Var.	4 Var.
·50	·8660	·8465	·7905	·7637
·51	·8680	·8493	·7937	·7681
·52	·8700	·8520	·7970	·7725
·53	·8720	·8548	·8002	·7769
·54	·8740	·8576	·8036	·7813
·55	·8760	·8605	·8070	·7858
·56	·8781	·8633	·8104	·7902
·57	·8802	·8662	·8140	·7947
·58	·8824	·8690	·8174	·7992
·59	·8846	·8720	·8210	·8037
·60	·8869	·8748	·8246	·8082
·61	·8892	·8777	·8282	·8128
·62	·8915	·8806	·8320	·8173
·63	·8938	·8835	·8357	·8220
·64	·8962	·8865	·8395	·8265
·65	·8986	·8894	·8433	·8311
·66	·9010	·8924	·8472	·8357
·67	·9034	·8954	·8511	·8404
·68	·9060	·8983	·8551	·8450
·69	·9084	·9013	·8590	·8497
·70	·9110	·9044	·8631	·8544
·71	·9136	·9074	·8672	·8590
·72	·9162	·9104	8713	·8637
·73	·9188	·9135	·8754	·8685
·74	·9215	·9166	·8796	·8732
·75	·9242	·9196	·8838	·8780
·76	·9270	·9227	·8881	·8827
·77	·9296	·9258	·8924	·8874
·78	·9324	·9290	·8967	·8922
·79	·9352	·9320	·9011	·8970
·80	·9380	·9352	·9055	·9018
·81	·9409	·9383	·9100	·9066
·82	·9438	·9415	·9144	·9114
·83	·9467	·9446	·9189	·9163
·84	·9496	·9478	·9234	·9211
·85	·9526	·9510	·9280	·9260
·86	·9556	·9542	·9326	·9308
·87	·9586	·9574	·9372	·9357
·88	·9616	·9606	·9419	·9406
·89	·9647	·9638	·9466	·9455
·90	·9678	·9671	·9513	·9504
·91	·9710	·9703	·9560	·9553
·92	·9740	·9736	·9608	·9602
·93	·9772	·9768	·9656	·9652
·94	·9804	·9801	·9704	·9701
·95	·9836	·9834	·9753	·9751
·96	·9868	·9867	·9802	·9800
·97	·9901	·9900	·9851	·9850
·98	·9933	·9933	·9900	·9900
·99	·9966	·9966	·9950	·9950
1·00	1·0000	1·0000	1·0000	1·0000

EXAMPLE.

Suppoſing the diameters to be 32 and 24, it is required to find the mean diameter for each variety.

Dividing 24 by 32 we obtain ·75, which being found in the column of quotients, oppoſite thereto ſtand the numbers

·9242 ·9196 ·8838 ·8780,	which being each multiplied by 32, produce reſpectively	29·5744 29·4272 28·2816 28·0960	the correſponding mean diameters required.

By the Sliding Rule.

Find the difference between the bung and head diameters upon the fourth face of the rule, or inſide of the third ſlider, and oppoſite thereto is, for each variety, a number to be added to the head diameter for the mean diameter required.

So in the above example, againſt 8, the difference of the diameters, are found the numbers

$$\left.\begin{array}{l}5\cdot60\\5\cdot10\\4\cdot56\\4\cdot12\end{array}\right\}\text{which being added to 24, there reſult}\left\{\begin{array}{l}29\cdot60\\29\cdot10\\28\cdot56\\28\cdot12\end{array}\right\}\text{for the reſpective mean diameters, all of which are too great except the 2d, which is too little.}$$

So that this method does not give the true mean diameter.

PROBLEM II.

To find the content of a caſk by the mean diameter on the ſliding rule.

RULE.

Set the length on *C* to the * gage point, 18·95 for ale, or 17·15 for wine, on *D*; then againſt the mean diameter on *D*, is the content on *C*.

EXAMPLE.

If the bung diameter be 32, the head 24, and the length 40 inches.

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* The above gage points are found thus, viz. $\sqrt{\frac{282}{\cdot785398\ \&c.}}$ = 18·95, and $\sqrt{\frac{231}{\cdot785398\ \&c.}}$ = 17·15.

Having found the mean diameters as in the laſt problem, and ſet 40 on C to 18·95 or 17·15 on D,

$$\text{againſt}\left\{\begin{matrix}29\cdot57\\29\cdot43\\28\cdot28\\28\cdot10\end{matrix}\right\}\text{on } D \text{ is}\left\{\begin{matrix}97\cdot4\\96\cdot5\\89\cdot1\\88\cdot0\end{matrix}\right\}\text{ or }\left\{\begin{matrix}119\cdot5\\118\\108\cdot8\\107\cdot3\end{matrix}\right\}$$

on C, as near as can be judged; which agree nearly with the contents determined in the preceding chapter.

CHAPTER V.

Of the gauging of all caſks in general by means of four dimenſions, viz. the length, the bung and head diameters, and the diameter taken in the middle between the bung and head.

GENERAL PROBLEM.

To find the content of any caſk in ale or wine gallons, by four dimenſions.

RULE.*

Add into one ſum the ſquare of the bung diameter, the ſquare of the head diameter, and 4 times the

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* This rule is taken from prop. 3 ſect. 1 part 4; the number ·0005$\frac{2}{3}$ being $= \frac{\cdot785398\ \&c.}{6\times231}$ extremely near, and $\frac{\cdot785398\ \&c.}{6\times282} =$ ·0004641844.

And with regard to the operation by the ſliding rule, it may be obſerved that 42 is $= \sqrt{\frac{6\times231}{\cdot785398\ \&c.}}$, and $46\cdot4 = \sqrt{\frac{6\times282}{\cdot785398\ \&c.}}$.

The above rule, by the ſaid prop. 3 ſect. 1 part 4, was proved to be accurately true, not only for all the four different varieties of caſks, but alſo for all caſks and ſolids generated from any conic ſection whatever; and although the caſk be not preciſely in the form of any ſuch curve, the rule will give the content extremely near the truth;

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the ſquare of the middle diameter between the bung and head, and multiply that ſum by the length of the caſk; then the product

$$\text{mult. by}\left\}\begin{array}{l}\cdot 00046418\\ \cdot 0005\frac{2}{3}\end{array}\right\}\left\{\begin{array}{l}\text{for ale}\\ \text{for wine}\end{array}\right\}\text{ will give the content.}$$

Note.

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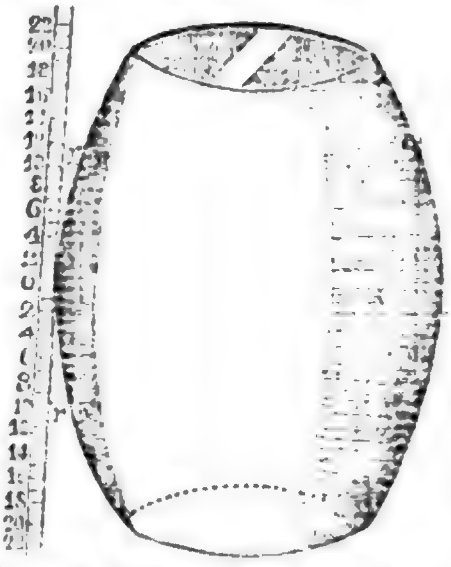
truth; ſo that whatever be the form of the caſk, we may in all caſes be pretty ſure of the content to within $\frac{1}{10}$ of a gallon, or perhaps leſs, ſuppoſing the dimenſions to be truly taken. So that more perfect than this, both with reſpect to truth and expedition, nothing can be expected or indeed wiſhed for in gauging: which makes me hope that one day this method will come into general uſe with the practitioners in the exciſe; and till then, I am fully perſuaded that much of their practice muſt be mere gueſſed work. The pretended difficulty of taking the middle diameter may perhaps deter many from uſing this method, but there cannot be any real difficulty in taking this diameter, except when a wooden hoop may happen to be on at the part where the diameter ought to be taken; but that will very rarely happen.

To find the fourth dimenſion, or diameter in the middle between the bung and head; upon one ſide of a ſquare rule let be drawn a ſcale of quarter inches, numbered both ways from nothing; and let the middle or 0 diviſion be applied to the bung or middle of the caſk, as in the margin, parallel or nearly parallel to the axe, and in the direction of the ſtaff; then whatever number of inches are in the whole length of the caſk, it is evident that, from the nature of the ſcale, the ſame diviſion or number of quarter inches will be oppoſite to the part where the middle diameter muſt be taken, which here ſuppoſe to be 10; and at 10, on each part of the rule, meaſure the diſtance *r s*; then if the ſum of *r s* and *r s* be taken from the bung diameter, there will remain the required middle diameter, excepting the allowance for the thickneſs of the ſtaff, which muſt be ſubducted.



Caſks

Note. To abbreviate the operation, $\frac{51}{11}$ may be uſed inſtead of 4·6418, to which it is nearly equal. Alſo

Caſks gauged by four dimenſions.

No.	Length	Head diam.	Bung diam.	Mid. diam.	Ale gallons	Diff. leſs than ſphr.
1	28·3	23·2	27·7	26·3	53·6	1·0
2	29·8	22·2	26·0	24·8	50·2	1·1
3	30·8	23·2	27·5	26·1	57·7	1·1
4	32·2	24·5	30·1	28·4	70·6	1·3
5	30·0	24·7	29·2	27·6	62·6	1·8
6	32·5	23·8	28·2	26·8	63·6	1·9
7	34·3	26·3	33·5	31·1	90·4	2·9
8	34·5	26·4	33·0	30·7	89·0	3·0
9	41·0	26·3	32·2	30·2	102·2	3·0
10	37·0	26·1	31·8	29·9	90·3	3·1
11	44·5	34·4	40·8	38·8	183·8	3·2
12	47·0	26·3	33·8	31·4	126·3	3·5
13	34·2	27·2	33·8	31·4	92·3	3·8
14	47·0	25·3	32·0	29·7	113·1	4·3
15	45·5	30·7	38·0	35·5	157·0	4·7
16	44·6	24·7	32·2	29·6	106·6	4·7
17	48·6	24·2	32·1	29·4	114·4	4·9
18	46·0	25·7	34·7	31·7	125·3	5·5
19	48·8	24·2	32·1	29·4	114·8	5·5
20	51·2	23·3	31·0	28·2	111·3	5·8
21	49·3	23·8	32·6	29·5	117·0	6·1
22	48·0	28·2	33·8	31·4	137·3	6·1
23	45·2	26·6	33·2	30·4	115·6	6·5
24	51·6	36·6	41·6	39·6	223·3	6·7
25	44·2	28·1	36·4	33·3	134·6	6·8
26	57·0	32·7	42·0	39·1	236·6	6·8
27	51·0	33·1	38·1	35·7	181·0	8·0
28	51·5	33·3	40·0	37·2	197·0	8·7
29	54·0	34·8	44·8	41·5	253·5	8·8
30	50·0	34·3	40·5	37·7	197·6	9·4
31	49·0	29·5	36·0	33·0	148·1	9·4
32	51·0	33·5	39·2	36·4	189·6	9·7
33	51·0	33·4	39·8	37·0	194·0	9·8
34	55·5	30·6	40·6	37·0	207·2	10·2
35	45·6	28·0	34·6	32·4	134·8	12·0
36	55·0	35·8	48·0	43·2	282·2	17·8

The annexed collection of caſks happened in real practice; and their dimenſions were carefully taken; but their contents were computed by a ſliding rule, and ſo may not all be preciſely true.

From hence it appears that a ſpheroidal caſk is a mere imaginary thing, the contents of real caſks being leſs than is aſſigned to them by that form; as indeed they ought, from the nature of the curves; for a ſpheroidal caſk would be leaſt curved in the middle and the moſt at the ends, whereas a real caſk is the leaſt curved at the ends, if it be any thing curved there at all; and indeed I have reaſon to think it is not, as will appear in chap. 7.

Alſo inſtead of 4 times the ſquare of the middle diameter, may be uſed the ſquare of twice that diameter.

BY THE SLIDING RULE.

Set the length on *C* to $\left\{\begin{matrix}46\cdot4 \text{ for ale}\\ 42\cdot0 \text{ for wine}\end{matrix}\right\}$ on *D*; and find both the bung, head, and middle diameters on *D*, noting the three numbers againſt them on *C*; then the ſum of the firſt and ſecond with 4 times the third will be the content required.

EXAMPLE.

Let the length of a caſk be 40 inches, the bung diameter 32, the head 24, and middle diameter 30·2 inches nearly $= \sqrt{912}$, which is taken upon the ſuppoſition that the caſk is ſpheroidal.

Then $\overline{32^2 + 24^2 + 4 \times 912} \times 40 \times \cdot0004641844 = 97\cdot44159$ &c. ale gallons; or multiplied by $\cdot0005\frac{2}{3}$, gives $118\cdot954\frac{2}{3}$ wine gallons for the content, the ſame as at prob. 1 chap. 3.

By the Sliding rule. Having ſet 40 on *C* to 46·4 on *D*, againſt 32, 24, and 30·2 on *D*, ſtand 19, 10·5, and 17 on *C*; then $19 + 10\cdot5 + 4 \times 17 = 97\cdot5$ ale gallons.

And, having ſet 40 on *C* to 42 on *D*, againſt 32, 24, and 30·2 on *D*, ſtand 23·2, 13, and 20·7 on *C*; then $23\cdot2 + 13 + 4 \times 20\cdot7 = 119$ wine gallons, for the content as before nearly.

CHAPTER VI.

A new, eaſy, and expeditious method of computing the content of a caſk from three dimenſions only, of whatever variety it may be.

RULE.*

In the column of diameters, or middle column, in the following table, find the bung and head diameters, taking out the number on the left of the head

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* The method of gauging here propoſed, is no other than the rule for one form of ſolids corrected ſo as to make it agree, in content, with the moſt common or general form of caſks: and, for the more expedition, the ſquares of all diameters, within certain limits, are divided by the proper conſtant diviſor, and the quotients ranged in the table oppoſite to their correſponding diameters; that in practice nothing more may be required than to multiply theſe quotients by any aſſigned length of a caſk, for the content in gallons.

Now it hath been found that wine hogſheads generally contain about a gallon and a quarter leſs than the content aſſigned to them by the rule for ſpheroidal caſks; to correct the rule for that form of caſks, then, we muſt increaſe the diviſor 882·355 in the proportion of 61¼ to 63; by which proportion we obtain 900, very nearly, to be uſed inſtead of 882·355, and then the rule will become $\frac{H^2+2B^2}{900}\times L$ for the content in gallons.

Then as the leaſt diameter of a wine hogſhead is about 19 inches, and the greateſt bung of a pipe about 35, to all diameters, in inches and tenths, from 18 to 36·3, I have computed the values of the quantities $\frac{H^2}{900}$ and $\frac{2B^2}{900}$, and diſpoſed them in the columns tituled head and bung areas; and which it is evident need only be taken out of the table, for any example of bung and head diameters, and multiplying their ſum by the length muſt give the content. So that we have here a general and eaſy rule by which any perſon, in half a minute, may compute the content of any caſk whoſe bung and head diameters, and length are given, and that generally nearer to the truth than by any rule now commonly uſed.

I ſhall here inſert the method in which I made this table; for although I have as above explained the property upon which the table

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head diameter, and the number on the right of the bung diameter; then the ſum of thoſe two numbers multiplied by the length of the caſk will give the content in wine gallons. And the wine gallons being multiplied by 77 and the product divided by 94, the quotient will be the ale gallons; or multiply by 9 and divide by 11 for ale gallons, becauſe $\frac{231}{282} = \frac{77}{94} = \frac{9}{11}$ extremely near. A

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table is founded, viz. that the numbers in it conſiſt of every diameter ſquared and then divided by 900, I did not find the numbers in that manner, but by the following method in perhaps a tenth part of the time required by the former.

Since the 1ſt, 2d, 3d, 4th, &c. numbers are

$$\frac{18^2}{900} = \cdot 36 = A,$$

$$\frac{18\cdot 1^2}{900} = \frac{18^2}{900} + \frac{3\cdot 6}{900} + \frac{\cdot 01}{900} = \frac{18^2}{900} + \frac{3\cdot 61}{900} = A + \frac{3\cdot 61}{900} = A + \cdot 0040\dot{1} = B,$$

$$\frac{18\cdot 2^2}{900} = \frac{18^2}{900} + \frac{2\times 3\cdot 6}{900} + \frac{\cdot 04}{900} = \frac{18^2}{900} + \frac{3\cdot 61}{900} + \frac{3\cdot 63}{900} = B + \frac{3\cdot 63}{900} = B + \cdot 0040\dot{3} = C,$$

$$\frac{18\cdot 3^2}{900} = \frac{18^2}{900} + \frac{3\times 3\cdot 6}{900} + \frac{\cdot 09}{900} = \frac{18^2}{900} + \frac{\cdot 06}{19.8} + \frac{3\cdot 63}{900} + \frac{3\cdot 65}{900} = C + \frac{3\cdot 65}{900} = C + \cdot 0040\dot{5} = D,$$

&c.

it is evident that the differences of the terms form an arithmetical ſeries whoſe common difference is $\frac{\cdot 02}{900} = \cdot 0000\dot{2}$, and therefore by writing down a column of differences, and to the firſt term adding the firſt difference, to obtain the ſecond term; then to the ſecond term adding the ſecond difference, for the third term; and ſo on as appears in the margin; the terms will evidently be found in a very expeditious manner: alſo by doubling theſe head areas we obtain the correſponding bung areas.——In the above computations the laſt figures, having a point above them, are infinite repetends.

Diff.	Areas.
	36000
401	36401
403	36804
405	37210
407	37617
410	38027
412	38440
414	38854

Farther, it is evident that in this method, the gauge point will be 30 (= √900), which is to be uſed, in the rule for ſpheroidal caſks, inſtead of 29·7, and the correct content will be obtained.

A,

A Table of Areas.

Head area	Diameter	Bung area	Head area	Diameter	Bung area	Head area	Diameter	Bung area	Head area	Diameter	Bung area
·3600	18·0	·7200	·5675	22·6	1·1350	·8220	27·2	1·6441	1·1236	31·8	2·2472
·3640	18·1	·7280	·5725	22·7	1·1451	·8281	27·3	1·6562	1·1307	31·9	2·2614
·3680	18·2	·7361	·5776	22·8	1·1552	·8342	27·4	1·6684	1·1378	32·0	2·2756
·3721	18·3	·7442	·5827	22·9	1·1654	·8403	27·5	1·6806	1·1449	32·1	2·2898
·3762	18·4	·7524	·5878	23·0	1·1756	·8464	27·6	1·6928	1·1520	32·2	2·3041
·3803	18·5	·7606	·5929	23·1	1·1858	·8525	27·7	1·7051	1·1592	32·3	2·3184
·3844	18·6	·7688	·5980	23·2	1·1961	·8587	27·8	1·7174	1·1664	32·4	2·3328
·3885	18·7	·7771	·6032	23·3	1·2064	·8649	27·9	1·7298	1·1736	32·5	2·3472
·3927	18·8	·7854	·6084	23·4	1·2168	·8711	28·0	1·7422	1·1808	32·6	2·3617
·3969	18·9	·7938	·6136	23·5	1·2272	·8773	28·1	1·7547	1·1881	32·7	2·3762
·4011	19·0	·8022	·6188	23·6	1·2377	·8836	28·2	1·7672	1·1954	32·8	2·3908
·4053	19·1	·8107	·6241	23·7	1·2482	·8899	28·3	1·7798	1·2027	32·9	2·4054
·4096	19·2	·8192	·6294	23·8	1·2588	·8962	28·4	1·7924	1·2100	33·0	2·4200
·4139	19·3	·8278	·6347	23·9	1·2694	·9025	28·5	1·8050	1·2173	33·1	2·4347
·4182	19·4	·8364	·6400	24·0	1·2300	·9088	28·6	1·8177	1·2247	33·2	2·4494
·4225	19·5	·8450	·6453	24·1	1·2907	·9152	28·7	1·8304	1·2321	33·3	2·4642
·4268	19·6	·8537	·6507	24·2	1·3014	·9216	28·8	1·8432	1·2395	33·4	2·4790
·4312	19·7	·8624	·6561	24·3	1·3122	·9280	28·9	1·8560	1·2469	33·5	2·4939
·4356	19·8	·8712	·6615	24·4	1·3230	·9344	29·0	1·8689	1·2544	33·6	2·5088
·4400	19·9	·8800	·6669	24·5	1·3339	·9409	29·1	1·8818	1·2619	33·7	2·5238
·4444	20·0	·8889	·6724	24·6	1·3448	·9474	29·2	1·8948	1·2694	33·8	2·5388
·4489	20·1	·8978	·6779	24·7	1·3558	·9539	29·3	1·9078	1·2769	33·9	2·5538
·4534	20·2	·9068	·6834	24·8	1·3668	·9604	29·4	1·9208	1·2844	34·0	2·5689
·4579	20·3	·9158	·6889	24·9	1·3778	·9669	29·5	1·9339	1·2920	34·1	2·5840
·4624	20·4	·9248	·6944	25·0	1·3888	·9735	29·6	1·9470	1·2996	34·2	2·5992
·4669	20·5	·9339	·7000	25·1	1·4000	·9801	29·7	1·9602	1·3072	34·3	2·6144
·4715	20·6	·9430	·7056	25·2	1·4112	·9867	29·8	1·9734	1·3148	34·4	2·6297
·4761	20·7	·9522	·7112	25·3	1·4224	·9933	29·9	1·9867	1·3225	34·5	2·6450
·4807	20·8	·9614	·7168	25·4	1·4337	1·0000	30·0	2·0000	1·3302	34·6	2·6604
·4853	20·9	·9707	·7225	25·5	1·4450	1·0067	30·1	2·0134	1·3379	34·7	2·6758
·4900	21·0	·9800	·7282	25·6	1·4564	1·0134	30·2	2·0268	1·3456	34·8	2·6912
·4947	21·1	·9894	·7339	25·7	1·4678	1·0201	30·3	2·0402	1·3533	34·9	2·7067
·4994	21·2	·9988	·7396	25·8	1·4792	1·0268	30·4	2·0537	1·3611	35·0	2·7222
·5041	21·3	1·0082	·7453	25·9	1·4907	1·0336	30·5	2·0772	1·3689	35·1	2·7378
·5088	21·4	1·0177	·7511	26·0	1·5022	1·0404	30·6	2·0808	1·3767	35·2	2·7534
·5136	21·5	1·0272	·7569	26·1	1·5138	1·0472	30·7	2·0944	1·3845	35·3	2·7691
·5184	21·6	1·0368	·7627	26·2	1·5254	1·0540	30·8	2·1081	1·3924	35·4	2·7848
·5232	21·7	1·0464	·7685	26·3	1·5371	1·0609	30·9	2·1218	1·4003	35·5	2·8006
·5280	21·8	1·0561	·7744	26·4	1·5488	1·0678	31·0	2·1356	1·4082	35·6	2·8164
·5329	21·9	1·0658	·7803	26·5	1·5606	1·0747	31·1	2·1494	1·4161	35·7	2·8322
·5378	22·0	1·0756	·7862	26·6	1·5724	1·0816	31·2	2·1632	1·4240	35·8	2·8481
·5427	22·1	1·0854	·7921	26·7	1·5842	1·0885	31·3	2·1771	1·4320	35·9	2·8640
·5476	22·2	1·0952	·7980	26·8	1·5961	1·0955	31·4	2·1910	1·4400	36·0	2·8800
·5525	22·3	1·1051	·8040	26·9	1·6080	1·1025	31·5	2·2050	1·4480	36·1	2·8960
·5575	22·4	1·1150	·8100	27·0	1·6200	1·1095	31·6	2·2190	1·4560	36·2	2·9121
·5625	22·5	1·1250	·8160	27·1	1·6320	1·1165	31·7	2·2331	1·4641	36·3	2·9282

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EXAMPLE.

Required the content of a caſk whoſe length is 40 inches, and bung and head diameters 32 and 24 inches.

To the right of 32, the bung diameter, is	2·2756 the bung area;
and to the left of 24, the head diameter, is	·6400 the head area,
the ſum	2·9156
multiplied by	40 the length
produces	116·624 wine gallons

for the content required.

To work by the ſliding rule, we may proceed as in prob. 1 chap. 3, uſing 30 as the gauge point inſtead of 29·7; thus, having ſet 40 on *C* to 30 on *D*,

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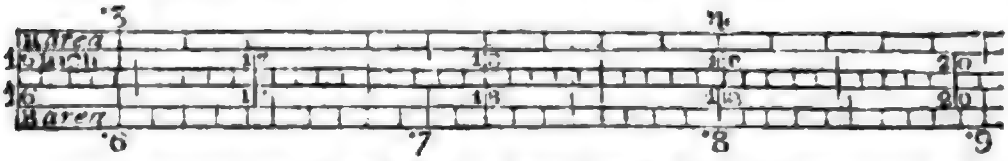
A ſtill farther improvement is to be made of this method, by tranſferring the table to ſcales upon a rod or the inſide of a Branan's rule inſtead of the two uſeleſs diagonal ſcales, a ſcale of inches and tenths correſponding to the ſcale of areas. A part of ſuch a ſcale is repreſented below.



And for the making of ſuch a ſcale I ſhall here ſubjoin a table of diameters computed from aſſumed equi-different areas, which is much fitter for this purpoſe than the foregoing table of areas which was computed from aſſumed equi-different diameters. This table of diameters was computed by *Mr.* AB. CROCKER, *Maſter of a boarding ſchool in Ilminſter, Somerſetſhire,* and who alſo communicated the ſubſtance of this chapter, together with the collection of caſks at the end of the laſt.

D, againſt 32 and 24 on *D* ſtand 45·6 and 25·5 on *C*; then $2 \times 45{\cdot}6 + 25{\cdot}5 = 91{\cdot}2 + 25{\cdot}5 = 116{\cdot}7$ wine gallons the content nearly as before.

And for the content in ale gallons, we have $116{\cdot}6 \times \frac{9}{11} = 95{\cdot}4$.

CHAP-

A table of diameters for the making of ſcales.

Head area	Diameter	Bung area	Head area	Diameter	Bung area	Head area	Diameter	Bung area	Head area	Diameter	Bung area
·30	16·43	·60	·58	22·85	1·16	·86	27·82	1·72	1·14	32·03	2·28
·31	16·70	·62	·59	23·04	1·18	·87	27·98	1·74	1·15	32·17	2·30
·32	16·97	·64	·60	23·24	1·20	·88	28·14	1·76	1·16	32·31	2·32
·33	17·24	·66	·61	23·43	1·22	·89	28·30	1·78	1·17	32·45	2·34
·34	17·50	·68	·62	23·62	1·24	·90	28·45	1·80	1·18	32·59	2·36
·35	17·76	·70	·63	23·81	1·26	·91	28·61	1·82	1·19	32·73	2·38
·36	18·00	·72	·64	24·00	1·28	·92	28·78	1·84	1·20	32·86	2·40
·37	18·26	·74	·65	24·18	1·30	·93	28·93	1·86	1·21	33·00	2·42
·38	18·50	·76	·66	24·37	1·32	·94	29·09	1·88	1·22	33·14	2·44
·39	18·73	·78	·67	24·56	1·34	·95	29·24	1·90	1·23	33·27	2·46
·40	18·97	·80	·68	24·74	1·36	·96	29·40	1·92	1·24	33·41	2·48
·41	19·21	·82	·69	24·92	1·38	·97	29·55	1·94	1·25	33·54	2·50
·42	19·44	·84	·70	25·10	1·40	·98	29·70	1·96	1·26	33·67	2·52
·43	19·67	·86	·71	25·28	1·42	·99	29·85	1·98	1·27	33·81	2·54
·44	19·90	·88	·72	25·45	1·44	1·00	30·00	2·00	1·28	33·94	2·56
·45	20·12	·90	·73	25·63	1·46	1·01	30·15	2·02	1·29	34·07	2·58
·46	20·35	·92	·74	25·81	1·48	1·02	30·30	2·04	1·30	34·20	2·60
·47	20·57	·94	·75	25·98	1·50	1·03	30·45	2·06	1·31	34·33	2·62
·48	20·79	·96	·76	26·15	1·52	1·04	30·60	2·08	1·32	34·47	2·64
·49	21·00	·98	·77	26·33	1·54	1·05	30·74	2·10	1·33	34·60	2·66
·50	21·21	1·00	·78	26·50	1·56	1·06	30·88	2·12	1·34	34·73	2·68
·51	21·42	1·02	·79	26·67	1·58	1·07	31·03	2·14	1·35	34·86	2·70
·52	21·63	1·04	·80	26·84	1·60	1·08	31·18	2·16	1·36	34·98	2·72
·53	21·84	1·06	·81	27·00	1·62	1·09	31·32	2·18	1·37	35·11	2·74
·54	22·05	1·08	·82	27·17	1·64	1·10	31·47	2·20	1·38	35·25	2·76
·55	22·25	1·10	·83	27·33	1·66	1·11	31·61	2·22	1·39	35·37	2·78
·56	22·45	1·12	·84	27·50	1·68	1·12	31·75	2·24	1·40	35·50	2·80
·57	22·65	1·14	·85	27·66	1·70	1·13	31·90	2·26	1·41	35·62	2·82

Now as to the method of forming the preceding table; ſince any area is equal to the ſquare of its diameter divided by 900, the diameter muſt be equal to the root of the product of 900 drawn into the

CHAPTER VII.

A new and very exact method of computing the content of a caſk of any form from three dimenſions only.

RULE.*

Add into one ſum 488 times the ſquare of the bung diameter, 313 times the ſquare of the head diameter,

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the area; ſo that to the firſt area ·30 will belong the diameter $\sqrt{\cdot 30\times 900}$ or $\sqrt{30\times 9} = \sqrt{A}$, to the 2d area ·31 the diameter $\sqrt{31\times 9} = \sqrt{A+9} = \sqrt{B}$, to the 3d area ·32 the diameter $\sqrt{32\times 9} = \sqrt{B+9} = \sqrt{C}$, &c. Hence it is evident that the ſucceſſive products, out of which the ſquare root is to be extracted, will be obtained by the continual addition of 9; and then extracting the roots, by means of the table of logarithms, will give the ſeveral diameters required.

But if you have a table of ſquare roots by you, ſince the root of 9 is 3, the moſt expeditious method of making the table, is to take the ſquare roots of 30, 31, 32, &c. out of the table, and then multiply them by 3 for the ſeveral correſponding diameters.

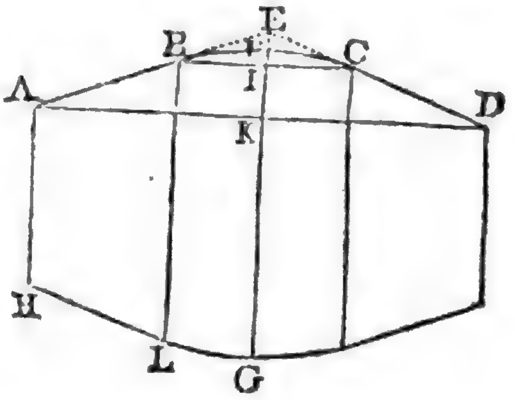
* This rule is taken from a method, of inveſtigating the contents of caſks, hinted by *Mr.* JAMES DAVIDSON, *Teacher of the Mathematics at Dundee in North Briton*, the invention of which take as below.

It appearing reaſonable that, in determining a proper rule for computing the contents of caſks, we ought to have regard to the general methods which are uſed by coopers in conſtructing them; application was therefore made to ſeveral ingenious workmen in that way, from whoſe account it appeared that, ordinarily, the ſides or edges of each ſtaff of a caſk, for about one-third of its length next each end, are made tapering in a ſtreight line, and that for the middle third part they were curved or made convex to form the bulge or middle of the caſk.

From this plain account then we are naturally led to conſider one-third of a caſk at each end as the fruſtum of a cone; and the middle part is moſt properly conſidered as in the form of a parabolic curve, for theſe reaſons; viz. 1ſt, that, on account of the ſhortneſs of this curved part, it may be conſidered but as a ſmall part near the vertex of the curve; 2d, that all curves near the vertex differ inſenſibly diameter, and 324 times the product of the diameters; multiply the ſum by the length, and the product by ·0000272; then the laſt product divided by 9 will give the wine gallons, and divided by 11 will give the ale gallons.

Ex-

ſibly from the parabola; and 3d, that its rules are moſt eaſily adapted to practical uſes. We ſhall therefore ſuppoſe the middle third part of a caſk to be the middle fruſtum of a parabolic ſpindle, and the ends as two fruſtums of a cone, and inveſtigate its contents thus.



Let AB and CD be the two right-lined parts, and BC the parabolic part; then produce AB and DC to meet in E, and draw the lines as are evident by the figure. Put the length AD of the caſk $= L$, the bung diameter $FG = B$, and the head diameter $AH = H$: then ſince AB hath the ſame direction as FB at B, ABE will be a tangent to the parabola BF, and therefore $FI = \frac{1}{2} IE$; but $BI = \frac{1}{3} AK$, and hence by ſimilar triangles $IE = \frac{1}{3} EK$; conſequently $FI = \frac{1}{2} IE = \frac{1}{6} EK = \frac{1}{5} KF = \frac{B-H}{10}$; ſo that the common diameter $BL = GF - 2FI = B - \frac{B-H}{5} = \frac{4B+H}{5}$, which call C.

Now, by the common rules for parabolic ſpindles and conic fruſtums, we obtain $\frac{8B^2 + 4BC + 3C^2}{15} \times \frac{Ln}{3} = \frac{328B^2 + 44BH + 3H^2}{25 \times 45} \times Ln$ for the parabolic or middle part, and $\frac{C^2 + CH + H^2}{3} \times \frac{2Ln}{3} = \frac{160B^2 + 280BH + 310H^2}{25 \times 45} \times Ln$ for the two ends; and the ſum of theſe two expreſſions will give $\frac{488B^2 + 324BH + 313H^2}{1125} \times Ln$ for the content in inches, n being $= ·785398$ &c. And the quantity

EXAMPLE.

Required the content of a caſk whoſe length is 40, bung diameter 32, and head diameter 24 inches.

Here $\overline{488 \times 32^2 + 324 \cdot 32 \times 24 + 313 \times 24^2} \times 40 \times \cdot0000272 = 1010\cdot569$, which being divided by 9 and by 11, we obtain 112·3 wine gallons, or 91·9 ale gallons for the content required.

CHAPTER VIII.

Of the Ullage of Caſks.

The ullage of a caſk is now generally underſtood to be either the empty part or the part filled, of a caſk which is not quite full.

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tity $\frac{\cdot785308\ \&c.}{1125}$ being divided by 231 gives $\cdot000003\frac{2}{90} = \frac{\cdot0000272}{9}$ the multiplier for wine gallons; and ſince 231 is to 282 as 9 to 11 extremely near, $\frac{\cdot0000272}{11}$ will be the multiplier for ale gallons, as in the rule.

And thus we have a rule which, although it be not the beſt accommodated to expedition, hath not only the property of agreeing with the content as computed from the method by four dimenſions when thoſe dimenſions are actually taken, but alſo agrees ſurprizingly well with the real contents of caſks; as hath been proved by ſeveral caſks which have actually been filled with a true gallon meaſure, after their contents have been computed by this method.

To ſhew its agreement with the four-dimenſion method, take any one of the examples out of the collection at the end of chapter 5, as for inſtance No. 14, whoſe content by that method is 113·1 gallons; and by this method it will be found to come out nearly 112. Again, if we take any other, as ſuppoſe the laſt whoſe content was 282·2, it will come out by this method 283·2. So then in theſe two inſtances, taken at random, although the contents be very large, they agree to within a gallon; the one method exceeding in the former caſe and the other in the latter.

In this buſineſs caſks are conſidered as in two poſitions, viz. with their axes either parallel or perpendicular to the horizon, that is, either lying along or ſtanding upright upon one end. The ullage of a caſk, either ſtanding or lying, may be determined for any particular variety by the preceding parts of this work; but the rules thence obtained are too complex for ordinary practice, eſpecially for a lying caſk; I ſhall not, therefore, introduce thoſe rules into this place, but ſhall content myſelf with putting down ſuch approximations as have been found to accord beſt with expedition and accuracy, together with the univerſal method by three diameters, which is not only ſufficiently eaſy but alſo the moſt accurate for this purpoſe.

PROBLEM I.

To find the ullage of a ſtanding and lying caſk by the lines SS *and* SL *on the ſliding rule.*

RULE.

By ſome of the preceding chapters, find the whole content of the caſk; then for a caſk

$$\left\{\begin{matrix}\text{ſtanding}\\ \text{lying}\end{matrix}\right\} \begin{matrix}\text{ſet}\\ \text{the}\end{matrix} \left\{\begin{matrix}\text{length}\\ \text{bung diam.}\end{matrix}\right\} \text{ on } N \text{ to 100 on } \left\{\begin{matrix}SS\\ SL\end{matrix}\right\},$$

and on the ſame $\left\{\begin{matrix}SS\\ SL\end{matrix}\right\}$ againſt the wet or dry inches on N is a number to be reſerved; then ſet 100 on B to the reſerved number on A, and againſt the whole content on B will be found the ullage on A; and this ullage will be either the empty part or the part filled, according as the dry or wet inches were uſed in finding the reſerved number.

Ex-

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EXAMPLE I.

Required the ullage of a ſtanding caſk whoſe length is 40, bung diameter 32, head diameter 24, wet inches 10, and, conſequently, dry inches 30 inches.

By the laſt chapter, the content was nearly 92 ale gallons: Hence, having ſet 40 on *N* to 100 on *SS*, and againſt 10 on *N* found 23 on *S*; I then ſet 100 on *B* to 23 on *A*, and againſt 92 on *B* find 21·2 ale gallons on *A*, for the quantity remaining in the caſk.

If 30 the dry inches be uſed, the reſerved ſegment will be found to be 76·7, and then the correſponding ullage is 70·2 for the gallons drawn off.

And the ſum of theſe two parts is 91·4 gallons, which is about half a gallon leſs than the whole content; which error would be inconſiderable if it were to be divided between the two parts; but inſtead of that generally the one part is too little, and the other too much, by which it happens that the error in each part commonly exceeds that of their ſum.

EXAMPLE II.

Let the dimenſions and content be the ſame in the caſe of the lying caſk, alſo the wet inches 8, and conſequently the dry inches 24.

Having ſet 32 on *N* to 100 on *SL*, againſt 8 and 24 on *N* are the reſerved ſegments 17·8 and 82·5 on *SL*; then having ſet 100 on *B* to 92 on *A*, againſt 17·8 and 82·5 on *B* are 16·4 and 76 on *A*, which are the parts filled and empty reſpectively, and whoſe ſum is 92·4, near half a gallon too much.

PROBLEM II.

To find the ullage of a ſtanding caſk by the pen.

RULE I.

Take here the rule in chapter 5, uſing the * diameter at the ſurface of the liquor inſtead of the bung diameter, the diameter in the middle between the ſurface of the liquor and its neareſt head inſtead of the middle diameter, and the diſtance between the ſurface and neareſt head inſtead of the length of the caſk; and you will obtain the part empty or filled according as the caſk is more or leſs than half full.

EXAMPLE.

Taking the above diameters 24, 27, and 29 inches, and the wet inches 10; we ſhall have $\overline{24^2 + 54^2 + 29^2} \times 10 \times \frac{51}{110000} = 4333 \times \frac{51}{11000} = 20{\cdot}9$ gallons for ullage required.

RULE II.

As the ſquare of the length of the caſk is to the ſquare of the difference between the ſaid length and wet

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[*] The diameters at the ſurface of the liquor and in the middle between it and the neareſt head, will eaſily be obtained by ſuſpending a plummet by a rule laid over the middle of the head, ſo as juſt to touch the bulge of the caſk, and then meaſuring the diſtance of the ſtring from the ſide of the caſk at the ſurface of the liquor and in the middle between it and the neareſt head; for then the doubles of theſe meaſures taken from the bung diameter will leave the ſaid required diameters.

wet or dry inches, viz. the leſs of them, ſo is the difference between the bung and head diameters to a number; which being taken from the bung diameter, will leave the diameter of a cylinder of the ſame length with, and nearly equal to the part filled or empty, viz. the leſs of them.*

EXAMPLE.

Taking the ſame example as in problem 1, we have as $40^2 : \overline{40-10}|^2 :: 32-24 : \frac{8 \times 30^2}{40^2} = \frac{8 \times 9}{16} = 4\frac{1}{2}$; and $32 - 4\frac{1}{2} = 27\frac{1}{2}$ the mean diameter.

Then $27{\cdot}5^2 \times 10 \times {\cdot}0027851 = 21$ ale gallons = the ullage required.

Note. The number ·0027851 is the conſtant multiplier $\frac{{\cdot}785398\ \&c.}{282}$.

PROBLEM III.

To find the ullage of a lying caſk by the pen.

RULE.

Divide the wet inches by the bung diameter, find the quotient in the firſt column of the table of cir-

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* This rule is founded upon the reaſonable ſuppoſition that for the ſmall part of a caſk, the diameter in the middle may be conſidered as a mean diameter; which is ſtrictly true when the caſk is paraboloidal, and exceedingly near the truth when parabolic-ſpindular; but the ſaid middle diameter, as given in the rule, is computed from the latter form, becauſe the rule is thereby rendered not only eaſier but generally more exact.

Thus, by the property of the parabola, $L^2 : \overline{L-l}|^2 :: B-H : \frac{B-H}{L^2} \times \overline{L-l}|^2$, and $B - \frac{B-H}{L^2} \times \overline{L-l}|^2 = M$; where B is the bung, H the head, and M the mean diameter, alſo L is the length of the caſk, and l the wet or dry inches.

cular ſegments (at the end of the work), and take out the ſegment ſtanding oppoſite to it, then multiply this ſegment by the whole content of the caſk, and then the product being either divided by ·785398 &c. or multiplied by 1·2732395 will give the ullage nearly.*

EXAMPLE.

Taking the ſame caſk as before, whoſe length is 40, bung diameter 32, head diameter 24; and ſuppoſing the wet inches to be 8.

By chapter 7, the whole content is nearly 92 ale gallons. Then $\frac{8}{32} = \frac{1}{4} = {}^{\cdot}25$; oppoſite to which, in the table of ſegments, is the ſegment ·15354621; hence $92 \times {}^{\cdot}15354621 \times 1{\cdot}2732395 = 18$ ale gallons = the ullage required.

SCHOLIUM.

Having delivered the neceſſary rules for meaſuring caſks, &c. I do not ſuppoſe that any thing more of the ſubject of gauging is neceſſary to be given in this book; for as to ciſterns, couches, &c. tuns, coolers, &c. coppers, ſtills, &c. which are firſt ſuppoſed to be in the form of ſome of the ſolids in the former parts of this work, and then meaſured accordingly, no perſon can be at a loſs concerning them who knows any thing of ſuch ſolids in general; and to treat of them here would induce me to a long

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* This method is evidently nothing elſe than taking the whole content, in ſuch proportion to the ullage as the whole bung circle bears to the ſegment of it cut off by the ſurface of the liquor; and is nearer to the truth than any other practical rule which I can find.

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a long and tedious repetition, only for the ſake of pointing out the proper multipliers, or diviſors, which is, I think, a reaſon very inadequate to ſo cumberſome an increaſe of my book.

I ſhall only juſt obſerve, that when tuns, &c. of oval baſes are to be gauged, as thoſe baſes really meaſure to more than true ellipſes of the ſame length and breadth, they ought to be meaſured by the equi-diſtant ordinate method delivered in ſection 2 of part 4.

And that when caſks are met with which have different head diameters, they may be deemed incompleat caſks, and their contents conſidered and meaſured as the ullage of a caſk.

SECTION III.

Of the works of Artificers.

THE artificers whoſe works are here to be treated of, are 1 Bricklayers, 2 Carpenters and Joiners, 3 Glaziers, 4 Maſons, 5 Painters, 6 Pavers, 7 Plaſterers, 8 Plumbers, and 9 Slaters and Tilers, &c.

Different branches of artificers eſtimate the value of their work by meaſures of different ſizes, ſuch as by the foot, the yard, the rod, the ſquare, &c. but they generally take the dimenſions in yards, or feet, inches and tenths or eighths, or feet, tenths, and hundredths; which laſt is certainly the beſt method of taking the dimenſions, becauſe that the computation will be performed by the common multiplication or by the ſliding rule deſcribed below. For taking meaſures, the moſt common inſtrument is what is called the carpenters rule, of which it will be neceſſary here to give a deſcription.

CHAPTER I.

Of the common or carpenters rule.

This inſtrument conſiſts of two pieces each of a foot in length, and connected together by a folding joint, and one of them having a ſlider running in it.

The one face is for taking dimenſions in inches and half-quarters or eighths, the whole length being divided into inches and eighths. Upon this face alſo are plain ſcales divided by diagonals into twelves, for planning dimenſions taken in feet and inches.

For taking dimenſions in feet, tenths, and hundredths, the edge of the rule is adapted, it being divided after that manner, viz. each foot into ten parts, and each of theſe tenths into ten parts again; that ſo the rule may ſuit the different inclinations of meaſurers, viz. thoſe who uſe twelves and thoſe who uſe tens; but as I prefer the latter method, for the reaſons above-mentioned, I would fain perſuade all meaſurers to do the ſame; and therefore in the following work I ſhall take the dimenſions and make the calculations in that manner as well as in feet, inches, and parts by croſs multiplication. For the duodecimal multiplication, or multiplication of feet, inches, and parts, runs all their notions of the parts of the products into confuſion, habituating them to account thoſe parts ſquare inches, &c. which are really quite different things.

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Upon one part of the other face are four lines marked *A*, *B*, *C*, and *D*; the two middle ones, *B* and *C*, being on a ſlider. Theſe four lines are logarithmic ones, the firſt three being all of the ſame radius, but the laſt of a radius double to that of the former; ſo that this line is but of one radius in length while each of the three former is of two; or the diviſions on it anſwer to the ſquare roots of the diviſions on the others; that is, when any number on the ſlider is ſet oppoſite to its root on *D*, then alſo are all the other numbers oppoſite to their ſquare roots: it is neceſſary to ſhift the ſlider out of its natural poſition to make theſe ſquares and roots correſpond, becauſe they begin with different numbers, *D* beginning with 4 and the reſt with the number 1. Upon the line *D* are marked *WG* and *AG*, the wine and ale gauge points, the former at 17·15 and the latter at 18·95, to make this inſtrument anſwer the end of a gauging rule. This line is alſo called and marked the *Girt Line*, becauſe it ſerves for the meaſuring of timber.

Upon the other part of this face is a table of the value of a load or 50 cubic feet of timber at all prices from 6 pence to 2 ſhillings a foot.

The particular uſes of this inſtrument will be ſeen in the following examples, where ſuch of them are wrought by the rule as were judged neceſſary to explain its uſe.

CHAPTER II.

Of the different meaſures uſed by different artificers.

144 ſquare inches = a ſquare foot,
9 ſquare feet = a ſquare yard,
63 ſquare feet = 7 ſquare yards = a rood,
100 ſquare feet = a ſquare,
$272\frac{1}{4}$ ſquare feet = $30\frac{1}{4}$ ſquare yards = a rod, perch, or ſquare pole.

So that if the content of any work be firſt computed in feet, which is generally the method, it will be found in yards, roods, ſquares, or rods, by dividing the feet by the correſponding diviſors 9, 63, 100 or $272\frac{1}{4}$.——Inſtead of the diviſor 63, divide firſt by 9 and then by 7, becauſe 9 times 7 make 63. You eaſily divide by 100, by only cutting off two figures on the right hand. To divide by $272\frac{1}{4}$, firſt multiply by 4 and then divide by 9, 11, and 11; becauſe $272\frac{1}{4} = \frac{9 \times 11 \times 11}{4}$. It may be noted here that this rod is the ſtatute perch uſed in land meaſuring, being the ſquare of the pole of $5\frac{1}{2}$ yards or $16\frac{1}{2}$ feet.

The above denominations are thoſe by which moſt kinds of work are valued; ſome particular articles indeed are valued by the foot running or lineal meaſure: By the ſquare foot are valued glazing and maſonry; by the ſquare yard, joinery, painting, paving, and plaſtering; by the ſquare, ſlating, tiling, and carpentry or floring, partitioning, and roofing; and by the rod, brick work. The rood is very little uſed except in ſome country places for very common ſtone

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ſtone walls: but it may be noted that this rood of 63 ſquare feet is not the rood uſed in land meaſuring; for this laſt rood is a quarter of an acre = 40 ſquare poles = 10890 ſquare feet.

A table for changing hundredth parts of feet into inches and parts, and the contrary; by means of which dimenſions taken in one of theſe may be readily changed to the other.

Centeſms, or hundredth parts of a foot	Inches.	12th pts.	Inches.	8th pts, or half quarters.	Centeſms.
1	0	1	0	1	1
2	0	3	0	2	2
3	0	4	0	3	3
4	0	6	0	4	4
5	0	7	0	5	5
6	0	9	0	6	6
7	0	10	0	7	7
8	1	0	1	0	8
9	1	1	2	0	17
10	1	2	3	0	25
20	2	5	4	0	33
30	3	7	5	0	42
40	4	10	6	0	50
50	6	0	7	0	58
60	7	2	8	0	67
70	8	5	9	0	75
80	9	7	10	0	83
90	10	10	11	0	92
100	12	0	12	0	100

EXAMPLE I.

If it be required to change 44 centeſms into inches and 12ths.——Againſt 4 and 40 in the firſt column ſtand reſpectively 6 parts and 4 in. 10 pts in the ſecond; then the ſum of theſe two is 5 inches 4 parts, the number required.

EXAMPLE II.

If it be required to change 5 inches and $\frac{3}{4}$ or 6 eighths to centeſms.——Againſt 6 eighths and 5 inches the third column ſtand reſpectively 6 and 42 in the laſt column; the ſum of which is 48 centeſms.

The reaſon why I have ſet 8ths in the third column, and not 12ths, is that the meaſurers who calculate by 12ths take the dimenſions in 8ths and change them into 12ths, becauſe that the inches upon the rule are divided into 8ths, and not 12ths.

Note. You may convert centeſms into inches and parts, and the contrary, without the table, by the following rule, viz.

Multiply centeſms by 12 for inches, cutting off two figures; and multiply theſe two figures by 12 again for parts, cutting off two figures here alſo. And divide 8ths or 12ths, with two cyphers affixed, by 8 or by 12 for inches; and divide inches by 12 for centeſms.

EXAMPLE I.

Thus, taking the firſt example above.

44 centeſms
12
Inches 5·28 That is 44 centeſms = 5 inches 3 parts.
12
Parts 3·36

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·1193	·05293105	·1238	·05587188	·1283	·05885898	·1328	·06189110
·1194	·05299589	·1239	·05593776	·1284	·05892588	·1329	·06195898
·1195	·05306075	·1240	·05600367	·1285	·05899279	·1330	·06202688
·1196	·05312564	·1241	·05606960	·1286	·05905973	·1331	·06209481
·1197	·05319055	·1242	·05613555	·1287	·05912670	·1332	·06216276
·1198	·05325548	·1243	·05620152	·1288	·05919368	·1333	·06223073
·1199	·05332044	·1244	·05626752	·1289	·05926069	·1334	·06229872
·1200	·05338542	·1245	·05633353	·1290	·05932772	·1335	·06236673
·1201	·05345042	·1246	·05639958	·1291	·05939477	·1336	·06243476
·1202	·05351545	·1247	·05646564	·1292	·05946184	·1337	·06250282
·1203	·05358050	·1248	·05653173	·1293	·05952894	·1338	·06257090
·1204	·05364558	·1249	·05659784	·1294	·05959605	·1339	·06263899
·1205	·05371067	·1250	·05666397	·1295	·05966319	·1340	·06270711
·1206	·05377579	·1251	·05673012	·1296	·05973035	·1341	·06277525
·1207	·05384094	·1252	·05679730	·1297	·05979754	·1342	·06284342
·1208	·05390610	·1253	·05686250	·1298	·05986474	·1343	·06291160
·1209	·05397130	·1254	·05692873	·1299	·05993197	·1344	·06297981
·1210	·05403651	·1255	·05699497	·1300	·05999922	·1345	·06304803
·1211	·05410175	·1256	·05706124	·1301	·06006649	·1346	·06311628
·1212	·05416701	·1257	·05712753	·1302	·06013379	·1347	·06318455
·1213	·05423229	·1258	·05719384	·1303	·06020110	·1348	·06325284
·1214	·05429760	·1259	·05726018	·1304	·06026844	·1349	·06332116
·1215	·05436293	·1260	·05732654	·1305	·06033580	·1350	·06338949
·1216	·05442828	·1261	·05739292	·1306	·06040318	·1351	·06345785
·1217	·05449366	·1262	·05745932	·1307	·06047058	·1352	·06352622
·1218	·05455906	·1263	·05752575	·1308	·06053801	·1353	·06359462
·1219	·05462448	·1264	·05759220	·1309	·06060546	·1354	·06366304
·1220	·05468992	·1265	·05765867	·1310	·06067293	·1355	·06373148
·1221	·05475539	·1266	·05772516	·1311	·06074042	·1356	·06379994
·1222	·05482088	·1267	·05779168	·1312	·06080793	·1357	·06386843
·1223	·05488640	·1268	·05785822	·1313	·06087546	·1358	·06393693
·1224	·05495194	·1269	·05792478	·1314	·06094302	·1359	·06400546
·1225	·05501750	·1270	·05799136	·1315	·06101060	·1360	·06407400
·1226	·05503308	·1271	·05805797	·1316	·06107820	·1361	·06414257
·1227	·05514869	·1272	·05812460	·1317	·06114582	·1362	·06421116
·1228	·05521432	·1273	·05819125	·1318	·06121347	·1363	·06427977
·1229	·05527997	·1274	·05825792	·1319	·06128113	·1364	·06434840
·1230	·05534565	·1275	·05832461	·1320	·06134882	·1365	·06441706
·1231	·05541135	·1276	·05839133	·1321	·06141653	·1366	·06448573
·1232	·05547707	·1277	·05845807	·1322	·06148426	·1367	·06455443
·1233	·05554281	·1278	·05852483	·1323	·06155201	·1368	·06462314
·1234	·05560858	·1279	·05859162	·1324	·06161978	·1369	·06469188
·1235	·05567437	·1280	·05865843	·1325	·06168758	·1370	·06476064
·1236	·05574019	·1281	·05872525	·1326	·06175540	·1371	·06482942
·1237	·05580602	·1282	·05879211	·1327	·06182324	·1372	·06489822

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·1373	·06496704	·1418	·06808567	·1463	·07124589	·1508	·07444665
·1374	·06503589	·1419	·06815545	·1464	·07131659	·1509	·07451823
·1375	·06510475	·1420	·06822525	·1465	·07138730	·1510	·07458983
·1376	·06517364	·1421	·06829507	·1466	·07145803	·1511	·07466145
·1377	·06524254	·1422	·06836491	·1467	·07152878	·1512	·07473309
·1378	·06531147	·1423	·06843478	·1468	·07159955	·1513	·07480475
·1379	·06538042	·1424	·06850466	·1469	·07167034	·1514	·07487643
·1380	·06544939	·1425	·06857456	·1470	·07174115	·1515	·07494812
·1381	·06551838	·1426	·06864448	·1471	·07181198	·1516	·07501984
·1382	·06558739	·1427	·06871443	·1472	·07188284	·1517	·07509158
·1383	·06565642	·1428	·06878439	·1473	·07195371	·1518	·07516333
·1384	·06572548	·1429	·06885437	·1474	·07202460	·1519	·07523511
·1385	·06579455	·1430	·06892438	·1475	·07209551	·1520	·07530690
·1386	·06586365	·1431	·06899440	·1476	·07216644	·1521	·07537872
·1387	·06593276	·1432	·06906445	·1477	·07223739	·1522	·07545055
·1388	·06600190	·1433	·06913451	·1478	·07230836	·1523	·07552240
·1389	·06607106	·1434	·06920460	·1479	·07237935	·1524	·07559427
·1390	·06614024	·1435	·06927470	·1480	·07245036	·1525	·07566616
·1391	·06620944	·1436	·06934483	·1481	·07252139	·1526	·07573808
·1392	·06627866	·1437	·06941498	·1482	·07259244	·1527	·07581001
·1393	·06634790	·1438	·06948515	·1483	·07266351	·1528	·07588195
·1394	·06641716	·1439	·06955533	·1484	·07273460	·1529	·07595392
·1395	·06648644	·1440	·06962554	·1485	·07280571	·1530	·07602591
·1396	·06655575	·1441	·06969577	·1486	·07287684	·1531	·07609792
·1397	·06662507	·1442	·06976602	·1487	·07294798	·1532	·07616994
·1398	·06669442	·1443	·06983629	·1488	·07301915	·1533	·07624199
·1399	·06676378	·1444	·06990657	·1489	·07309034	·1534	·07631406
·1400	·06683317	·1445	·06997688	·1490	·07316155	·1535	·07638614
·1401	·06690258	·1446	·07004721	·1491	·07323278	·1536	·07645824
·1402	·06697201	·1447	·07011756	·1492	·07330402	·1537	·07653037
·1403	·06704146	·1448	·07018793	·1493	·07337529	·1538	·07660251
·1404	·06711093	·1449	·07025832	·1494	·07344658	·1539	·07667467
·1405	·06718042	·1450	·07032873	·1495	·07351788	·1540	·07674685
·1406	·06724993	·1451	·07039916	·1496	·07358921	·1541	·07681905
·1407	·06731946	·1452	·07046961	·1497	·07366056	·1542	·07689127
·1408	·06738901	·1453	·07054008	·1498	·07373192	·1543	·07696350
·1409	·06745859	·1454	·07061057	·1499	·07380331	·1544	·07703576
·1410	·06752818	·1455	·07068108	·1500	·07387471	·1545	·07710804
·1411	·06759780	·1456	·07075162	·1501	·07394613	·1546	·07718033
·1412	·06766743	·1457	·07082217	·1502	·07401758	·1547	·07725265
·1413	·06773709	·1458	·07089274	·1503	·07408904	·1548	·07732498
·1414	·06780676	·1459	·07096333	·1504	·07416052	·1549	·07739733
·1415	·06787646	·1460	·07103394	·1505	·07423203	·1550	·07746970
·1416	·06794618	·1461	·07110457	·1506	·07430355	·1551	·07754209
·1417	·06801592	·1462	·07117522	·1507	·07437509	·1552	·07761450

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Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·1553	·07768693	·1598	·08096576	·1643	·08428219	·1688	·08763532
·1554	·07775938	·1599	·08103905	·1644	·08435631	·1689	·08771024
·1555	·07783185	·1600	·08111236	·1645	·08443044	·1690	·08778518
·1556	·07790433	·1601	·08118569	·1646	·08450460	·1691	·08786014
·1557	·07797684	·1602	·08125904	·1647	·08457877	·1692	·08793512
·1558	·07804936	·1603	·08133241	·1648	·08465296	·1693	·08801011
·1559	·07812190	·1604	·08140580	·1649	·08472717	·1694	·08808512
·1560	·07819446	·1605	·08147920	·1650	·08480140	·1695	·08816015
·1561	·07826704	·1606	·08155262	·1651	·08487564	·1696	·08823520
·1562	·07833964	·1607	·08162607	·1652	·08494991	·1697	·08831027
·1563	·07841226	·1608	·08169953	·1653	·08502419	·1698	·08838535
·1564	·07848490	·1609	·08177300	·1654	·08509849	·1699	·08846045
·1565	·07855756	·1610	·08184650	·1655	·08517281	·1700	·08853557
·1566	·07863023	·1611	·08192002	·1656	·08524714	·1701	·08861070
·1567	·07870292	·1612	·08199355	·1657	·08532149	·1702	·08868585
·1568	·07877564	·1613	·08206710	·1658	·08539587	·1703	·08876103
·1569	·07884837	·1614	·08214067	·1659	·08547025	·1704	·08883621
·1570	·07892112	·1615	·08221426	·1660	·08554466	·1705	·08891142
·1571	·07899389	·1616	·08228789	·1661	·08561909	·1706	·08898664
·1572	·07906668	·1617	·08236150	·1662	·08569353	·1707	·08906188
·1573	·07913949	·1618	·08243514	·1663	·08576799	·1708	·08913714
·1574	·07921231	·1619	·08250880	·1664	·08584247	·1709	·08921242
·1575	·07928516	·1620	·08258248	·1665	·08591697	·1710	·08928771
·1576	·07935802	·1621	·08265618	·1666	·08599148	·1711	·08936302
·1577	·07943090	·1622	·08272990	·1667	·08606601	·1712	·08943835
·1578	·07950380	·1623	·08280364	·1668	·08614056	·1713	·08951369
·1579	·07957672	·1624	·08287739	·1669	·08621513	·1714	·08958906
·1580	·07964966	·1625	·08295116	·1670	·08628972	·1715	·08966444
·1581	·07972262	·1626	·08302495	·1671	·08636432	·1716	·08973983
·1582	·07979560	·1627	·08309876	·1672	·08643894	·1717	·08981525
·1583	·07986859	·1628	·08317259	·1673	·08651358	·1718	·08989068
·1584	·07994161	·1629	·08324644	·1674	·08658824	·1719	·08996613
·1585	·08001464	·1630	·08332030	·1675	·08666292	·1720	·09004160
·1586	·08008769	·1631	·08339418	·1676	·08673761	·1721	·09011709
·1587	·08016076	·1632	·08346808	·1677	·08681232	·1722	·09019259
·1588	·08023385	·1633	·08354200	·1678	·08688705	·1723	·09026811
·1589	·08030696	·1634	·08361594	·1679	·08696180	·1724	·09034364
·1590	·08038008	·1635	·08368989	·1680	·08703656	·1725	·09041920
·1591	·08045323	·1636	·08376387	·1681	·08711134	·1726	·09049477
·1592	·08052639	·1637	·08383786	·1682	·08718614	·1727	·09057036
·1593	·08059957	·1638	·08391187	·1683	·08726096	·1728	·09064596
·1594	·08067277	·1639	·08398590	·1684	·08733580	·1729	·09072159
·1595	·08074599	·1640	·08405994	·1685	·08741065	·1730	·09079723
·1596	·08081923	·1641	·08413401	·1686	·08748552	·1731	·09087289
·1597	·08089248	·1642	·08420809	·1687	·08756041	·1732	·09094856

Vert. ſine	Seg. area	Vert. ſine	Seg. area	Vert. ſine	Seg. area	Vert. ſine	Seg. area
·1733	·09102425	·1778	·09444815	·1823	[illegible]	·1868	·10139754
·1734	·09109996	·1779	·09452463	·1824	[illegible]	·1869	·10147549
·1735	·09117569	·1780	·09460112	·1825	[illegible]	·1870	·10155347
·1736	·09125143	·1781	·09467764	·1826	[illegible]	·1871	·10103146
·1737	·09132720	·1782	·09475416	·1827	[illegible]	·1872	·10170947
·1738	·09140298	·1783	·09483071	·1828	[illegible]	·1873	·10178749
·1739	·09147877	·1784	·09490727	·1829	[illegible]	·1874	·10186553
·1740	·09155458	·1785	·09498385	·1830	[illegible]	·1875	·10194359
·1741	·09163042	·1786	·09506044	·1831	[illegible]	·1876	·10202165
·1742	·09170626	·1787	·09513705	·1832	[illegible]	·1877	·10209974
·1743	·09178213	·1788	·09521367	·1833	[illegible]	·1878	·10217784
·1744	·09185801	·1789	·09529033	·1834	[illegible]	·1879	·10225596
·1745	·09193391	·1790	·09536699	·1835	[illegible]	·1880	·10233409
·1746	·09200983	·1791	·09544367	·1836	[illegible]	·1881	·10241224
·1747	·09208576	·1792	·09552036	·1837	[illegible]	·1882	·10249041
·1748	·09216171	·1793	·09559708	·1838	·09906630	·1883	·10256859
·1749	·09223768	·1794	·09567381	·1839	·09914377	·1884	·10264679
·1750	·09231366	·1795	·09575055	·1840	·09922126	·1885	·10272500
·1751	·09238966	·1796	·09582731	·1841	·09929877	·1886	·10280323
·1752	·09246568	·1797	·09590409	·1842	·09937629	·1887	·10288148
·1753	·09254172	·1798	·09598089	·1843	·09945383	·1888	·10295974
·1754	·09261777	·1799	·09605770	·1844	·09953138	·1889	·10303802
·1755	·09269384	·1800	·09613453	·1845	·09960895	·1890	·10311631
·1756	·00276993	·1801	·09621138	·1846	·09968654	·1891	·10319462
·1757	·09284603	·1802	·09628824	·1847	·09976414	·1892	·10327295
·1758	·09292216	·1803	·09636512	·1848	·09984176	·1893	·10335129
·1759	·09299829	·1804	·09644201	·1849	·09991939	·1894	·10342965
·1760	·09307445	·1805	·09651893	·1850	·09999704	·1895	·10350802
·1761	·09315062	·1806	·09659585	·1851	·10007471	·1896	·10358641
·1762	·09322681	·1807	·09667280	·1852	·10015240	·1897	·10366481
·1763	·09330302	·1808	·09674976	·1853	·10023010	·1898	·10374324
·1764	·09337924	·1809	·09682674	·1854	·10030781	·1899	·10382167
·1765	·09345548	·1810	·09690374	·1855	·10038555	·1900	·10390013
·1766	·09353174	·1811	·09698075	·1856	·10046330	·1901	·10397859
·1767	·09360802	·1812	·09705778	·1857	·10054106	·1902	·10405708
·1768	·09368431	·1813	·09713482	·1858	·10061884	·1903	·10413558
·1769	·09376062	·1814	·09721188	·1859	·10069664	·1904	·10421409
·1770	·09383694	·1815	·09728896	·1860	·10077445	·1905	·10429262
·1771	·09391328	·1816	·09736606	·1861	·10085228	·1906	·10437117
·1772	·09398964	·1817	·09744317	·1862	·10093012	·1907	·10444973
·1773	·09406602	·1818	·09752029	·1863	·10100799	·1908	·10452831
·1774	·09414241	·1819	·09759744	·1864	·10108586	·1909	·10460691
·1775	·09421882	·1820	·09767460	·1865	·10116376	·1910	·10468552
·1776	·09429525	·1821	·09775178	·1866	·10124167	·1911	·10476414
·1777	·09437169	·1822	·09782897	·1867	·10131959	·1912	·10484278

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·1913	·10492144	·1958	·10847713	·2003	·11206387	·2048	·11568094
·1914	·10500011	·1959	·10855650	·2004	·11214392	·2049	·11576166
·1915	·10507880	·1960	·10863589	·2005	·11222399	·2050	·11584239
·1916	·10515751	·1961	·10871529	·2006	·11230407	·2051	·11592314
·1917	·10523623	·1962	·10879471	·2007	·11238417	·2052	·11600390
·1918	·10531496	·1963	·10887414	·2008	·11246428	·2053	·11608468
·1919	·10539371	·1964	·10895359	·2009	·11254441	·2054	·11616547
·1920	·10547248	·1965	·10903305	·2010	·11262455	·2055	·11624628
·1921	·10555126	·1966	·10911253	·2011	·11270471	·2056	·11632710
·1922	·10563006	·1967	·10919202	·2012	·11278488	·2057	·11640793
·1923	·10570887	·1968	·10927153	·2013	·11286507	·2058	·11648878
·1924	·10578770	·1969	·10935105	·2014	·11294527	·2059	·11656965
·1925	·10586655	·1970	·10943059	·2015	·11302549	·2060	·11665053
·1926	·10594541	·1971	·10951014	·2016	·11310572	·2061	·11673142
·1927	·10602428	·1972	·10958971	·2017	·11318597	·2062	·11681233
·1928	·10610318	·1973	·10966930	·2018	·11326623	·2063	·11689325
·1929	·10618208	·1974	·10974890	·2019	·11334650	·2064	·11697419
·1930	·10626101	·1975	·10982851	·2020	·11342679	·2065	·11705514
·1931	·10633994	·1976	·10990814	·2021	·11350710	·2066	·11713610
·1932	·10641890	·1977	·10998779	·2022	·11358742	·2067	·11721708
·1933	·10649787	·1978	·11006745	·2023	·11366776	·2068	·11729808
·1934	·10657685	·1979	·11014713	·2024	·11374811	·2069	·11737909
·1935	·10665585	·1980	·11022682	·2025	·11382847	·2070	·11746011
·1936	·10673487	·1981	·11030652	·2026	·11390885	·2071	·11754115
·1937	·10681390	·1982	·11038624	·2027	·11398925	·2072	·11762220
·1938	·10689295	·1983	·11046598	·2028	·11406966	·2073	·11770327
·1939	·10697201	·1984	·11054573	·2029	·11415008	·2074	·11778435
·1940	·10705109	·1985	·11062550	·2030	·11423052	·2075	·11786545
·1941	·10713018	·1986	·11070528	·2031	·11431097	·2076	·11794656
·1942	·10720929	·1987	·11078507	·2032	·11439144	·2077	·11802768
·1943	·10728842	·1988	·11086489	·2033	·11447193	·2078	·11810882
·1944	·10736756	·1989	·11094471	·2034	·11455242	·2079	·11818998
·1945	·10744671	·1990	·11102456	·2035	·11463294	·2080	·11827114
·1946	·10752588	·1991	·11110441	·2036	·11471346	·2081	·11835233
·1947	·10760507	·1992	·11118429	·2037	·11479401	·2082	·11843352
·1948	·10768427	·1993	·11126417	·2038	·11487456	·2083	·11851474
·1949	·10776349	·1994	·11134407	·2039	·11495514	·2084	·11859596
·1950	·10784272	·1995	·11142399	·2040	·11503572	·2085	·11867720
·1951	·10792197	·1996	·11150392	·2041	·11511632	·2086	·11875846
·1952	·10800123	·1997	·11158387	·2042	·11519694	·2087	·11883972
·1953	·10808051	·1998	·11166383	·2043	·11527757	·2088	·11892101
·1954	·10815980	·1999	·11174381	·2044	·11535821	·2089	·11900230
·1955	·10823911	·2000	·11182380	·2045	·11543887	·2090	·11908362
·1956	·10831844	·2001	·11190381	·2046	·11551955	·2091	·11916494
·1957	·10839778	·2002	·11198383	·2047	·11560024	·2092	·11924628

Verſ. fine	Seg. area	Verſ. fine	Seg. area	Verſ. fine	Seg. area	Verſ. fine	Seg. area
·2093	·11932764	·2138	·12300320	·2183	·12670719	·2228	·13043872
·2094	·11940901	·2139	·12308523	·2184	·12678981	·2229	·13052195
·2095	11949039	·2140	·12316730	·2185	·12687245	·2230	·13060520
·2096	11957179	·2141	·12324933	·2186	·12695511	·2231	·13068846
·2097	·11965320	·2142	·12333138	·2187	·12703777	·2232	·13077173
·2098	·11973462	·2143	·12341344	·2188	·12712045	·2233	·13085501
·2099	·11981606	·2144	·12349551	·2189	·12720314	·2234	·13093831
·2100	·11989752	·2145	·12357760	·2190	·12728585	·2235	·13102162
·2101	·11997899	·2146	·12365970	·2191	·12736857	·2236	·13110495
·2102	·12006047	·2147	·12374182	·2192	·12745131	·2237	·13118828
·2103	·12014197	·2148	·12382395	·2193	·12753405	·2238	·13127164
·2104	·12022348	·2149	·12390609	·2194	·12761682	·2239	·13135500
·2105	·12030501	·2150	·12398825	·2195	·12769959	·2240	·13143838
·2106	·12038655	·2151	·12407042	·2196	·12778238	·2241	·13152177
·2107	·12046810	·2152	·12415261	·2197	·12786518	·2242	·13160517
·2108	·12054967	·2153	·12423481	·2198	·12794800	·2243	·13168859
·2109	·12063125	·2154	·12431702	·2199	·12803082	·2244	·13177202
·2110	·12071285	·2155	·12439924	·2200	·12811367	·2245	·13185546
·2111	·12079446	·2156	·12448149	·2201	·12819652	·2246	·13193892
·2112	·12087608	·2157	·12456374	·2202	·12827939	·2247	·13202239
·2113	·12095772	·2158	·12464601	·2203	·12836228	·2248	·13210588
·2114	·12103938	·2159	·12472829	·2204	·12844517	·2249	·13218937
·2115	·12112104	·2160	·12481059	·2205	·12852808	·2250	·13227288
·2116	·12120272	·2161	·12489290	·2206	·12861101	·2251	·13235641
·2117	·12128442	·2162	·12497522	·2207	·12869394	·2252	·13243994
·2118	·12136613	·2163	·12505756	·2208	·12877689	·2253	·13252349
·2119	·12144785	·2164	·12513991	·2209	·12885986	·2254	·13260705
·2120	·12152959	·2165	·12522227	·2210	·12894283	·2255	·13269063
·2121	·12161134	·2166	·12530465	·2211	·12902583	·2256	·13277422
·2122	·12169311	·2167	·12538704	·2212	·12910883	·2257	·13285782
·2123	·12177489	·2168	·12546945	·2213	·12919185	·2258	·13294143
·2124	·12185668	·2169	·12555187	·2214	·12927488	·2259	·13302506
·2125	·12193849	·2170	·12563430	·2215	·12935792	·2260	·13310870
·2126	·12202031	·2171	·12571675	·2216	·12944098	·2261	·13319236
·2127	·12210215	·2172	·12579921	·2217	·12952405	·2262	·13327603
·2128	·12218400	·2173	·12588169	·2218	·12960714	·2263	·13335971
·2129	·12226587	·2174	·12596418	·2219	·12969024	·2264	·13344340
·2130	·12234774	·2175	·12604668	·2220	·12977335	·2265	·13352711
·2131	·12242964	·2176	·12612919	·2221	·12985647	·2266	·13361083
·2132	·12251154	·2177	·12621172	·2222	·12993961	·2267	·13369456
·2133	·12259346	·2178	·12629427	·2223	·13002276	·2268	·13377831
·2134	·12267540	·2179	·12637682	·2224	·13010593	·2269	·13386206
·2135	·12275735	·2180	·12645939	·2225	·13018911	·2270	·13394584
·2136	·12283931	·2181	·12654198	·2226	·13027230	·2271	·13402962
·2137	·12292129	·2182	·12662458	·2227	·13035550	·2272	·13411342

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Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·237	·13419723	·2318	.13798210	2363	·14179270	·2408	·14562845
·2274	·13428105	·2319	·13806650	·2364	·14187767	·2409	·14571397
·2275	·13436489	·2320	·13815091	2365	·14196265	·2410	·14579950
·2276	·13444874	·2321	·13823534	·2366	·14204764	·2411	·14588505
·2277	·13453260	·2322	·13831978	·2367	·14213265	·2412	·14597060
·2278	·13461648	·2323	·13840424	2368	·14221767	·2413	·14605617
·2279	·13470037	·2324	·13848870	·2369	·14230270	·2414	·14614175
·2280	·13478427	·2325	·13857318	·2370	·14238774	·2415	·14622735
·2281	·13486819	·2326	·13865767	·2371	·14247279	·2416	·14631295
·2282	·13495212	·2327	·13874218	·2372	·14255786	·2417	·14639857
·2283	·13503606	·2328	·13882669	·2373	·14264294	·2418	·14648419
·2284	·13512001	·2329	·13891123	·2374	·14272803	·2419	·14656984
·2285	·13520398	·2330	·13899577	·2375	·14281314	2420	·14665549
·2286	·13528796	·2331	·13908032	·2376	·14289825	·2421	·14674115
·2287	·13537195	·2332	·14916489	·2377	·14298338	·2422	·14682683
·2288	·13545595	·2333	·14924947	·2378	·14306852	·2423	·14691252
·2289	·13553997	·2334	·14933406	·2379	·14315368	·2424	·14699822
·2290	·13562400	·2335	·14941867	·2380	·14323884	·2425	·14708393
·2291	·13570805	·2336	·14950328	·2381	·14332402	·2426	·14716966
·2292	·13579211	·2337	·14958791	·2382	·14340921	·2427	·14725540
·2293	·13587618	·2338	·14967256	·2383	·14349441	·2428	·14734114
·2294	·13596026	·2339	·14975721	·2384	·14357963	·2429	·14742691
·2295	·13604435	·2340	·14984188	·2385	·14366485	·2430	·14751268
·2296	·13612846	·2341	·14992656	·2386	·14375009	·2431	·14759846
·2297	·13621259	·2342	·14001126	·2387	·14383535	·2432	·14768426
·2298	·13629672	·2343	·14009596	·2388	·14392061	·2433	·14777007
·2299	·13638087	·2344	·14018068	·2389	·14400589	·2434	·14785589
·2300	·13646503	·2345	·14026541	·2390	·14409117	·2435	·14794172
·2301	·13654920	·2346	·14035015	·2391	·14417647	·2436	·14802757
·2302	·13663339	·2347	·14043491	·2392	·14426179	·2437	·14811342
·2303	·13671758	·2348	·14051968	·2393	·14434711	·2438	·14819929
·2304	·13680180	·2349	·14060446	·2394	·14443245	·2439	·14828517
·2305	·13688602	·2350	·14068925	·2395	·14451780	·2440	·14837107
·2306	·13697026	·2351	·14077406	·2396	·14460316	·2441	·14845697
·2307	·13705451	·2352	·14085888	·2397	·14468854	·2442	·14854289
·2308	·13713877	·2353	·14094371	·2398	·14477392	·2443	·14862882
·2309	·13722304	·2354	·14102855	·2399	·14485932	·2444	·14871476
·2310	·13730733	·2355	·14111341	·2400	·14494473	·2445	·14880071
·2311	·13739163	·2356	·14119828	·2401	·14503015	·2446	·14888667
·2312	·13747595	·2357	·14128316	·2402	·14511559	·2447	·14897265
·2313	·13756027	·2358	·14136805	·2403	·14520103	·2448	·14905864
·2314	·13764461	·2359	·14145296	·2404	·14528649	·2449	·14914464
·2315	·13772896	·2360	·14153787	·2405	·14537197	·2450	·14923065
·2316	·13781333	·2361	·14162281	·2406	·14545745	·2451	·14931667
·2317	·13789771	·2362	·14170775	·2407	·14554294	·2452	·14940271

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·2453	·14948875	·2498	·15337303	·2543	·15728071	·2588	·16121125
·2454	·14957481	·2499	·15345961	·2544	·15736781	·2589	·16129885
·2455	·14966088	·2500	·15354621	·2545	·15745492	·2590	·16138646
·2456	·14974696	·2501	·15363282	·2546	·15754205	·2591	·16147409
·2457	·14983306	·2502	·15371944	·2547	·15762918	·2592	·16156172
·2458	·14991917	·2503	·15380607	·2548	·15771632	·2593	·16164937
·2459	·15000528	·2504	·15389271	·2549	·15780348	·2594	·16173702
·2460	·15009141	·2505	·15397937	·2550	·15789064	·2595	·16182469
·2461	·15017755	·2506	·15406603	·2551	·15797782	·2596	·16191237
·2462	·15026371	·2507	·15415271	·2552	·15806501	·2597	·16200005
·2463	·15034987	·2508	·15423940	·2553	·15815221	·2598	·16208775
·2464	·15043605	·2509	·15432610	·2554	·15823942	·2599	·16217546
·2465	·15052224	·2510	·15441281	·2555	·15832665	·2600	·16226319
·2466	·15060844	·2511	·15449954	·2556	·15841388	·2601	·16235092
·2467	·15069465	·2512	·15458627	·2557	·15850113	·2602	·16243866
·2468	·15078088	·2513	·15467302	·2558	·15858838	·2603	·16252641
·2469	·15086711	·2514	·15475978	·2559	·15867565	·2604	·16261418
·2470	·15095336	·2515	·15484655	·2560	·15876293	·2605	·16270196
·2471	·15103962	·2516	·15493333	·2561	·15885022	·2606	·16278974
·2472	·15112589	·2517	·15502012	·2562	·15893752	·2607	·16287754
·2473	·15121217	·2518	·15510692	·2563	·15902483	·2608	·16296535
·2474	·15129847	·2519	·15519374	·2564	·15911215	·2609	·16305317
·2475	·15138477	·2520	·15528056	·2565	·15919949	·2610	·16314100
·2476	·15147109	·2521	·15536740	·2566	·15928684	·2611	·16322884
·2477	·15155742	·2522	·15545425	·2567	·15937419	·2612	·16331669
·2478	·15164376	·2523	·15554111	·2568	·15946157	·2613	·16340456
·2479	·15173011	·2524	·15562798	·2569	·15954894	·2614	·16349243
·2480	·15181648	·2525	·15571487	·2570	·15963633	·2615	·16358032
·2481	·15190285	·2526	·15580176	·2571	·15972373	·2616	·16366821
·2482	·15198924	·2527	·15588867	·2572	·15981114	·2617	·16375612
·2483	·15207564	·2528	·15597559	·2573	·15989856	·2618	·16384404
·2484	·15216205	·2529	·15606252	·2574	·15998600	·2619	·16393196
·2485	·15224848	·2530	·15614946	·2575	·16007345	·2620	·16401990
·2486	·15233491	·2531	·15623641	·2576	·16016091	·2621	·16410785
·2487	·15242136	·2532	·15632337	·2577	·16024837	·2622	·16419581
·2488	·15250781	·2533	·15641035	·2578	·16033585	·2623	·16428379
·2489	·15259428	·2534	·15649733	·2579	·16042334	·2624	·16437177
·2490	·15268077	·2535	·15658433	·2580	·16051084	·2625	·16445976
·2491	·15276726	·2536	·15667134	·2581	·16059836	·2626	·16454777
·2492	·15285376	·2537	·15675836	·2582	·16068588	·2627	·16463578
·2493	·15294028	·2538	·15684539	·2583	·16077341	·2628	·16472381
·2494	·15302681	·2539	·15693243	·2584	·16086096	·2629	·16481184
·2495	·15311334	·2540	·15701949	·2585	·16094852	·2630	·16489989
·2496	·15319989	·2541	·15710655	·2586	·16103608	·2631	·16498795
·2497	·15328646	·2542	·15719363	·2587	·16112366	·2632	·16507602

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Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·2633	·16516410	·2678	·16913870	·2723	·17313455	·2768	·17715111
·2634	·16525219	·2679	·16922727	·2724	·17322359	·2769	·17724060
·2635	·16534029	·2680	·16931585	·2725	·17331263	·2770	·17733010
·2636	·16542840	·2681	·16940444	·2726	·17340168	·2771	·17741961
·2637	·16551652	·2682	·16949304	·2727	·17349075	·2772	·17750913
·2638	·16560465	·2683	·16958165	·2728	·17357983	·2773	·17759865
·2639	·16569280	·2684	·16967027	·2729	·17366891	·2774	·17768819
·2640	·16578095	·2685	·16975890	·2730	·17375800	·2775	·17777774
·2641	·16586912	·2686	·16984754	·2731	·17384711	·2776	·17786730
·2642	·16595729	·2687	·16993619	·2732	·17393622	·2777	17795687
·2643	·16604548	·2688	·17002485	·2733	·17402535	·2778	·17804644
·2644	·16613368	·2689	·17011352	·2734	·17411448	·2779	·17813603
·2645	·16622188	·2690	·17020221	·2735	·17420363	·2780	·17822563
·2646	·16631010	·2691	·17029090	·2736	·17429278	·2781	·17831524
·2647	·16639833	·2692	·17037960	·2737	·17438195	·2782	·17840486
·2648	·16648657	·2693	·17046832	·2738	·17447113	·2783	·17849448
·2649	·16657482	·2694	·17055704	·2739	·17456031	·2784	·17858412
·2650	·16666308	·2695	·17064578	·2740	·17464951	·2785	·17867377
·2651	16675136	·2696	·17073452	·2741	·17473872	·2786	·17876342
·2652	·16683964	·2697	·17082328	·2742	·17482793	·2787	·17885309
·2653	·16692793	·2698	·17091204	·2743	·17491716	·2788	·17894277
·2654	·16701624	·2699	·17100082	·2744	·17500640	·2789	·17903245
·2655	·16710455	·2700	·17108961	·2745	·17509565	·2790	·17912215
·2656	·16719287	·2701	·17117840	·2746	·17518490	·2791	·17921186
·2657	·16728121	·2702	·17126721	·2747	·17527417	·2792	·17930157
·2658	·16736956	·2703	·17135603	·2748	·17536345	·2793	·17939130
·2659	·16745791	·2704	·17144486	·2749	·17545274	·2794	·17948104
·2660	·16754628	·2705	·17153370	·2750	·17554203	·2795	·17957079
·2661	·16763466	·2706	·17162254	·2751	·17563134	·2796	·17966054
·2662	·16772305	·2707	·17171140	·2752	·17572066	·2797	·17975030
·2663	·16781145	·2708	·17180027	·2753	·17580999	·2798	·17984008
·2664	·16789986	·2709	·17188915	·2754	·17589933	·2799	·17992986
·2665	·16798828	·2710	·17197804	·2755	·17598867	·2800	·18001966
·2666	·16807671	·2711	·17206694	·2756	·17607803	·2801	·18010946
·2667	·16816515	·2712	·17215585	·2757	·17616740	·2802	·18019928
·2668	·16825360	·2713	·17224477	·2758	·17625678	·2803	·18028910
·2669	·16834207	·2714	·17233371	·2759	·17634617	·2804	·18037894
·2670	·16843054	·2715	·17242265	·2760	·17643556	·2805	·18046878
·2671	·16851902	·2716	·17251160	·2761	·17652497	·2806	·18055863
·2672	·16860752	·2717	·17260056	·2762	·17661439	·2807	·18064850
·2673	·16869602	·2718	·17268953	·2763	·17670382	·2808	·18073837
·2674	·16878454	·2719	·17277852	·2764	·17679326	·2809	·18082825
·2675	·16887306	·2720	·17286751	·2765	·17688271	·2810	·18091815
·2676	·16896160	·2721	·17295651	·2766	·17697217	·2811	·18100805
·2677	·16905015	·2722	·17304553	·2767	·17706163	·2812	·18109796

Ver.. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·2813	·18118788	·2858	13524435	·2903	·18932002	·2948	·19341440
·2814	·18127781	·2859	·18533471	·2904	·18941080	·2949	·19350560
·2815	·18136775	·2860	·18542509	·2905	·18950160	·2950	·19359680
·2816	·18145771	·2861	·18551547	·2906	·18959240	·2951	·19368801
·2817	·18154767	·2862	·18560586	·2907	·18968321	·2952	·19377924
·2818	·18163764	·2863	·18569626	·2908	·18977404	·2953	·19387047
·2819	·18172762	·2864	·18578667	·2909	·18986487	·2954	·19396171
·2820	·14181761	·2865	·18587709	·2910	·18995571	·2955	·19405296
·2821	·18190761	·2866	·18596752	·2911	·19004656	·2956	·19414421
·2822	·18199762	·2867	·18605796	·2912	·19013741	·2957	·19423548
·2823	·18208763	·2868	·18614841	·2913	·19022828	·2958	·19432676
·2824	·18217766	·2869	·18623887	·2914	·19031916	·2959	·19441804
·2825	·18226770	·2870	·18632934	·2915	·19041005	·2960	·19450933
·2826	·18235775	·2871	·18641981	·2916	·19050094	·2961	·19460064
·2827	·18244781	·2872	·18651030	·2917	·19059184	·2962	·19469195
·2828	·18253787	·2873	·18660080	·2918	·19068276	·2963	·19478327
·2829	·18262795	·2874	·18669130	·2919	·19077368	·2964	·19487460
·2830	·18271804	·2875	·18678182	·2920	·19086461	·2965	·19496594
·2831	·18280813	·2876	·18687234	·2921	·19095555	·2966	·19505728
·2832	·18289824	·2877	·18696287	·2922	·19104650	·2967	·19514864
·2833	·18298835	·2878	·18705342	·2923	·19113746	·2968	·19524001
·2834	·18307848	·2879	·18714397	·2924	·19122843	·2969	·19533138
·2835	·18316861	·2880	·18723453	·2925	·19131941	·2970	·19542276
·2836	·18325876	·2881	·18732510	·2926	·19141040	·2971	·19551415
·2837	·18334891	·2882	·18741568	·2927	·19150139	·2972	·19560556
·2838	·18343907	·2883	·18750627	·2928	·19159240	·2973	·19569696
·2839	·18352925	·2884	·18759687	·2929	·19168341	·2974	·19578838
·2840	·18361943	·2885	·18768748	·2930	·19177443	·2975	·19587981
·2841	·18370962	·2886	·18777810	·2931	·19186547	·2976	·19597125
·2842	·18379982	·2887	·18786872	·2932	·19195651	·2977	·19606269
·2843	·18389004	·2888	·18795936	·2933	·19204756	·2978	·19615415
·2844	·18398026	·2889	·18805001	·2934	·19213862	·2979	·19624561
·2845	·18407049	·2890	·18814066	·2935	·19222969	·2980	·19633708
·2846	·18416073	·2891	·18823132	·2936	·19232076	·2981	·19642856
·2847	·18425098	·2892	·18832200	·2937	·19241185	·2982	·19652005
·2848	·18434124	·2893	·18841268	·2938	·19250295	·2983	·19661155
·2849	·18443150	·2894	·18850337	·2939	·19259405	·2984	·19670305
·2850	·18452178	·2895	·18859407	·2940	·19268517	·2985	·19679457
·2851	·18461207	·2896	·18868479	·2941	·19277629	·2986	·19688609
·2852	·18470237	·2897	·18877551	·2942	·19286742	·2987	·19697763
·2853	·18479267	·2898	·18886623	·2943	·19295856	·2988	·19706917
·2854	·18488299	·2899	·18895697	·2944	·19304971	·2989	·19716072
·2855	·18497332	·2900	·18904772	·2945	·19314087	·2990	·19725228
·2856	·18506365	·2901	·18913848	·2946	·19323204	·2991	·19734385
·2857	·18515400	·2902	·18922924	·2947	·19332321	·2992	·19743542

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Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·2993	·19752701	·3038	·20165737	·3083	·20580501	·3128	·20996946
·2994	·19761861	·3039	·20174935	·3084	·20589737	·3129	·21006219
·2995	·19771021	·3040	·20184134	·3085	·20598974	·3130	·21015493
·2996	·19780182	·3041	·20193335	·3086	·20608212	·3131	·21024767
·2997	·19789344	·3042	·20202535	·3087	·20617451	·3132	·21034043
·2998	·19798507	·3043	·20211737	·3088	·20626690	·3133	·21043319
·2999	·19807671	·3044	·20220940	·3089	·20635930	·3134	·21052596
·3000	·19816836	·3045	·20230143	·3090	·20645172	·3135	·21061874
·3001	·19826001	·3046	·20239348	·3091	·20654414	·3136	·21071153
·3002	·19835168	·3047	·20248553	·3092	·20663657	·3137	·21080432
·3003	·19844335	·3048	·20257759	·3093	·20672900	·3138	·21089713
·3004	·19853503	·3049	·20266966	·3094	·20682145	·3139	·21098994
·3005	·19862672	·3050	·20276173	·3095	·20691390	·3140	·21108276
·3006	·19871842	·3051	·20285382	·3096	·20700636	·3141	·21117558
·3007	·19881013	·3052	·20294591	·3097	·20709883	·3142	·21126842
·3008	·19890185	·3053	·20303802	·3098	·20719131	·3143	·21136126
·3009	·19899357	·3054	·20313013	·3099	·20728380	·3144	·21145411
·3010	·19908531	·3055	·20322225	·3100	·20737629	·3145	·21154697
·3011	·19917705	·3056	·20331438	·3101	·20746879	·3146	·21163984
·3012	·19926880	·3057	·20340651	·3102	·20756131	·3147	·21173272
·3013	·19936056	·3058	·20349866	·3103	·20765382	·3148	·21182560
·3014	·19945233	·3059	·20359081	·3104	·20774635	·3149	·21191849
·3015	·19954411	·3060	·20368297	·3105	·20783889	·3150	·21201139
·3016	·19963589	·3061	·20377514	·3106	·20793143	·3151	·21210430
·3017	·19972769	·3062	·20386732	·3107	·20802398	·3152	·21219721
·3018	·19981949	·3063	·20395951	·3108	·20811654	·3153	·21229013
·3019	·19991130	·3064	·20405170	·3109	·20820911	·3154	·21238307
·3020	·20000313	·3065	·20414391	·3110	·20830169	·3155	·21247600
·3021	·20009495	·3066	·20423612	·3111	·20839427	·3156	·21256895
·3022	·20018679	·3067	·20432834	·3112	·20848687	·3157	·21266191
·3023	·20027864	·3068	·20442057	·3113	·20857947	·3158	·21275487
·3024	·20037049	·3069	·20451281	·3114	·20867208	·3159	·21284784
·3025	·20046236	·3070	·20460505	·3115	·20876469	·3160	·21294082
·3026	·20055423	·3071	·20469731	·3116	·20885732	·3161	·21303380
·3027	·20064611	·3072	·20478957	·3117	·20894995	·3162	·21312680
·3028	·20073800	·3073	·20488184	·3118	·20904259	·3163	·21321980
·3029	·20082990	·3074	·20497412	·3119	·20913524	·3164	·21331281
·3030	·20092181	·3075	·20506641	·3120	·20922790	·3165	·21340583
·3031	·20101372	·3076	·20515870	·3121	·20932057	·3166	·21349886
·3032	·20110565	·3077	·20525101	·3122	·20941324	·3167	·21359189
·3033	·20119758	·3078	·20534332	·3123	·20950592	·3168	·21368493
·3034	·20128952	·3079	·20543564	·3124	·20959861	·3169	·21377798
·3035	·20138147	·3080	·20552797	·3125	·20969131	·3170	·21387104
·3036	·20147343	·3081	·20562031	·3126	·20978402	·3171	·21396410
·3037	·20156539	·3082	·20571265	·3127	·20987673	·3172	·21405718

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·3173	·21415026	·3218	·21834696	·3263	·22255911	·3308	·22678627
·3174	·21424335	·3219	·21844040	·3264	22265289	3309	·22688037
·3175	·21433644	3220	·21853384	·3265	22274667	·3310	·22697448
·3176	·21442955	·3221	·21862729	·3266	·22284046	·3311	·22706860
·3177	·21452266	·3222	·21872076	·3267	·22293426	3312	·22716273
·3178	·21461578	·3223	·21881422	·3268	·22302806	·3313	·22725686
·3179	·21470891	3224	·21890770	·3269	·22312187	·3314	·22735100
·3180	·21480205	·3225	·21900118	·3270	·22321569	·3315	·22744514
·3181	·21489519	·3226	·21909467	·3271	22330952	3316	·22753930
·3182	·21498834	·3227	·21918817	·3272	·22340336	·3317	·22763346
·3183	·21508150	·3228	·21928168	·3273	·22349720	·3318	·22772763
·3184	·21517467	·3229	·21937519	·3274	·22359105	·3319	·22782180
·3185	·21526784	·3230	·21946871	·3275	·22368490	3320	·22791599
·3186	·21536103	·3231	·21956224	·3276	·22377877	·3321	·22801018
·3187	·21545422	·3232	·21965577	·3277	·22387264	·3322	·22810437
·3188	·21554742	·3233	·21974932	·3278	·22396652	·3323	·22819858
·3189	·21564062	·3234	·21984287	·3279	·22406040	·3324	·22829279
·3190	·21573384	·3235	·21993643	·3280	·22415430	·3325	·22838701
·3191	·21582706	·3236	·22002999	·3281	·22424820	·3326	·22848123
·3192	·21592029	·3237	·22012357	·3282	·22434211	·3327	·22857546
·3193	·21601352	·3238	·22021715	·3283	·22443602	·3328	·22866970
·3194	·21610677	·3239	·22031074	·3284	·22452994	·3329	·22876395
·3195	·21620002	·3240	·22040433	·3285	·22462387	·3330	·22885820
·3196	·21629328	·3241	·22049794	·3286	·22471781	·3331	·22895246
·3197	·21638655	·3242	·22059155	·3287	·22481175	·3332	·22904673
·3198	·21647983	·3243	·22068517	·3288	·22490571	·3333	·22914101
·3199	·21657311	·3244	·22077879	·3289	·22499966	·3334	·22923529
·3200	·21666640	·3245	·22087243	·3290	·22509363	·3335	·22932958
·3201	·21675970	·3246	·22096607	·3291	·22518761	·3336	·22942388
·3202	·21685301	·3247	·22105972	·3292	·22528159	·3337	·22951818
·3203	·21694632	·3248	·22115337	·3293	·22537557	·3338	·22961249
·3204	·21703964	·3249	·22124704	·3294	·22546957	·3339	·22970681
·3205	·21713297	·3250	·22134071	·3295	·22556357	·3340	·22980113
·3206	·21722631	·3251	·22143439	·3296	·22565758	·3341	·22989546
·3207	·21731966	3252	·22152807	·3297	·22575160	·3342	·22998980
·3208	·21741301	·3253	·22162177	·3298	·22584562	·3343	·23008415
·3209	·21750637	·3254	·22171547	·3299	·22593966	·3344	·23017850
·3210	·21759974	·3255	·22180918	·3300	·22603370	·3345	·23027286
·3211	·21769311	·3256	·22190289	·3301	·22612774	·3346	·23036722
·3212	·21778650	·3257	·22199661	·3302	·22622179	·3347	·23046160
·3213	·21787989	·3258	·22209035	·3303	·22631586	·3348	·23055598
·3214	·21797329	·3259	·22218408	·3304	22640992	·3349	·23065037
·3215	·21806669	·3260	·22227783	·3305	·22650400	·3350	·23074476
·3216	·21816011	·3261	·22237158	·3306	·22659808	·3351	·23083916
·3217	·21825353	·3262	·22246534	·3307	·22669217	·3352	·23093357

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Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·3353	·23102799	·3398	·23528384	·3443	·23955339	·3488	·24383621
·3354	·23112241	·3399	·23537857	·3444	·23964842	·3489	·24393153
·3355	·23121684	·3400	·23547331	·3445	·23974345	·3490	·24402685
·3356	·23131128	·3401	·23556805	·3446	·23983850	·3491	·24412219
·3357	·23140572	·3402	·23566280	·3447	·23993355	·3492	·24421753
·3358	·23150017	·3403	·23575756	·3448	·24002861	·3493	·24431288
·3359	·23159463	·3404	·23585233	·3449	·24012367	·3494	·24440823
·3360	·23168909	·3405	·23594710	·3450	·24021874	·3495	·24450359
·3361	·23178356	·3406	·23604188	·3451	·24031382	·3496	·24459895
·3362	·23187804	·3407	·23613666	·3452	·24040890	·3497	·24469432
·3363	·23197253	·3408	·23623146	·3453	·24050399	·3498	·24478970
·3364	·23206702	·3409	·23632626	·3454	·24059909	·3499	·24488509
·3365	·23216152	·3410	·23642106	·3455	·24069419	·3500	·24498048
·3366	·23225602	·3411	·23651587	·3456	·24078930	·3501	·24507588
·3367	·23235054	·3412	·23661069	·3457	·24088442	·3502	·24517128
·3368	·23244505	·3413	·23670552	·3458	·24097954	·3503	·24526669
·3369	·23253958	·3414	·23680035	·3459	·24107467	·3504	·24536210
·3370	·23263411	·3415	·23689519	·3460	·24116980	·3505	·24545753
·3371	·23272866	·3416	·23699004	·3461	·24126494	·3506	·24555295
·3372	·23282320	·3417	·23708489	·3462	·24136009	·3507	·24564839
·3373	·23291776	·3418	·23717975	·3463	·24145525	·3508	·24574383
·3374	·23301232	·3419	·23727461	·3464	·24155041	·3509	·24583928
·3375	·23310689	·3420	·23736949	·3465	·24164558	·3510	·24593473
·3376	·23320146	·3421	·23746437	·3466	·24174075	·3511	·24603019
·3377	·23329604	·3422	·23755925	·3467	·24183593	·3512	·24612566
·3378	·23339063	·3423	·23765414	·3468	·24193112	·3513	·24622113
·3379	·23348523	·3424	·23774904	·3469	·24202631	·3514	·24631661
·3380	·23357983	·3425	·23784395	·3470	·24212151	·3515	·24641209
·3381	·23367444	·3426	·23793886	·3471	·24221672	·3516	·24650758
·3382	·23376905	·3427	·23803378	·3472	·24231193	·3517	·24660308
·3383	·23386368	·3428	·23812871	·3473	·24240715	·3518	·24669858
·3384	·23395831	·3429	·23822364	·3474	·24250238	·3519	·24679409
·3385	·23405294	·3430	·23831858	·3475	·24259761	·3520	·24688961
·3386	·23414759	·3431	·23841352	·3476	·24269285	·3521	·24698513
·3387	·23424224	·3432	·23850848	·3477	·24278809	·3522	·24708066
·3388	·23433689	·3433	·23860343	·3478	·24288334	·3523	·24717619
·3389	·23443156	·3434	·23869840	·3479	·24297860	·3524	·24727173
·3390	·23452623	·3435	·23879337	·3480	·24307386	·3525	·24736728
·3391	·23462090	·3436	·23888835	·3481	·24316913	·3526	·24746283
·3392	·23471559	·3437	·23898334	·3482	·24326441	·3527	·24755839
·3393	·23481028	·3438	·23907833	·3483	·24335969	·3528	·24765396
·3394	·23490498	·3439	·23917333	·3484	·24345498	·3529	·24774953
·3395	·23499968	·3440	·23926833	·3485	·24355028	·3530	·24784511
·3396	·23509439	·3441	·23936334	·3486	·24364558	·3531	·24794069
·3397	·23518911	·3442	·23945836	·3487	·24374089	·3532	·24803628

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·3533	·24813187	·3578	·25243997	·3623	·25676009	·3668	·26109180
·3534	·24822748	·3579	·25253585	·3624	·25685622	·3669	·26118819
·3535	·24832308	·3580	·25263173	·3625	·25695237	·3670	·26128459
·3536	·24841870	·3581	·25272761	·3626	·25704851	·3671	·26138099
·3537	·24851432	·3582	·25282350	·3627	·25714467	·3672	·26147739
·3538	·24860995	·3583	·25291940	·3628	·25724082	·3673	·26157381
·3539	·24870558	·3584	·25301530	·3629	·25733699	·3674	·26167022
·3540	·24880122	·3585	·25311121	·3630	·25743316	·3675	·26176664
·3541	·24889686	·3586	·25320713	·3631	·25752933	·3676	·26186307
·3542	·24899251	·3587	·25330305	·3632	·25762552	·3677	·26195950
·3543	·24908817	·3588	·25339898	·3633	·25772170	·3678	·26205594
·3544	·24918383	·3589	·25349491	·3634	·25781790	·3679	·26215239
·3545	·24927950	·3590	·25359085	·3635	·25791410	·3680	·26224884
·3546	·24937518	·3591	·25368679	·3636	·25801030	·3681	·26234529
·3547	·24947086	·3592	·25378274	·3637	·25810651	·3682	·26244175
·3548	·24956655	·3593	·25387870	·3638	·25820272	·3683	·26253822
·3549	·24966224	·3594	·25397466	·3639	·25829895	·3684	·26263469
·3550	·24975794	·3595	·25407063	·3640	·25839517	·3685	·26273117
·3551	·24985365	·3596	·25416660	·3641	·25849141	·3686	·26282765
·3552	·24994936	·3597	·25426258	·3642	·25858764	·3687	·26292414
·3553	·25004508	·3598	·25435857	·3643	·25868389	·3688	·26302063
·3554	·25014080	·3599	·25445456	·3644	·25878014	·3689	·26311713
·3555	·25023653	·3600	·25455055	·3645	·25887639	·3690	·26321363
·3556	·25033227	·3601	·25464656	·3646	·25897265	·3691	·26331014
·3557	·25042801	·3602	·25474257	·3647	·25906892	·3692	·26340666
·3558	·25052376	·3603	·25483858	·3648	·25916519	·3693	·26350318
·3559	·25061951	·3604	·25493460	·3649	·25926147	·3694	·26359970
·3560	·25071527	·3605	·25503063	·3650	·25935775	·3695	·26369623
·3561	·25081104	·3606	·25512666	·3651	·25945404	·3696	·26379277
·3562	·25090681	·3607	·25522270	·3652	·25955034	·3697	·26388931
·3563	·25100259	·3608	·25531874	·3653	·25964664	·3698	·26398586
·3564	·25109837	·3609	·25541479	·3654	·25974294	·3699	·26408241
·3565	·25119416	·3610	·25551085	·3655	·25983925	·3700	·26417897
·3566	·25128996	·3611	·25560691	·3656	·25993557	·3701	·26427553
·3567	·25138576	·3612	·25570297	·3657	·26003189	·3702	·26437210
·3568	·25148157	·3613	·25579905	·3658	·26012822	·3703	·26446868
·3569	·25157738	·3614	·25589512	·3659	·26022455	·3704	·26456526
·3570	·25167320	·3615	·25599121	·3660	·26032089	·3705	·26466184
·3571	·25176903	·3616	·25608730	·3661	·26041724	·3706	·26475843
·3572	·25186486	·3617	·25618339	·3662	·26051359	·3707	·26485503
·3573	·25196070	·3618	·25627950	·3663	·26060994	·3708	·26495163
·3574	·25205654	·3619	·25637560	·3664	·26070630	·3709	·26504824
·3575	·25215239	·3620	·25647172	·3665	·26080267	·3710	·26514485
·3576	·25224824	·3621	·25656783	·3666	·26089904	·3711	·26524147
·3577	·25234411	·3622	·25666396	·3667	·26099542	·3712	·26533809

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Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·3713	·26543472	·3758	·26978841	·3803	·27415259	·3848	·27852657
·3714	·26553135	·3759	·26988528	·3804	·27424959	·3849	·27862388
·3715	·26562799	·3760	·26998216	·3805	·27434669	·3850	·27872120
·3716	·26572463	·3761	·27007903	·3806	·27444380	·3851	·27881852
·3717	·26582128	·3762	·27017592	·3807	·27454091	·3852	·27891585
·3718	·26591793	·3763	·27027281	·3808	·27463802	·3853	·27901318
·3719	·26601459	·3764	·27036970	·3809	·27473514	·3854	·27911051
·3720	·26611125	·3765	·27046660	·3810	·27483226	·3855	·27920785
·3721	·26620792	·3766	·27056350	·3811	·27492939	·3856	·27930520
·3722	·26630460	·3767	·27066041	·3812	·27502653	·3857	·27940255
·3723	·26640128	·3768	·27075733	·3813	·27512367	·3858	·27949990
·3724	·26649796	·3769	·27085425	·3814	·27522081	·3859	·27959726
·3725	·26659465	·3770	·27095117	·3815	·27531796	·3860	·27969463
·3726	·26669135	·3771	·27104810	·3816	·27541511	·3861	·27979199
·3727	·26678805	·3772	·27114503	·3817	·27551227	·3862	·27988937
·3728	·26688476	·3773	·27124198	·3818	·27560943	·3863	·27998675
·3729	·26698147	·3774	·27133892	·3819	·27570660	·3864	·28008413
·3730	·26707818	·3775	·27143587	·3820	·27580377	·3865	·28018152
·3731	·26717491	·3776	·27153282	·3821	·27590095	·3866	·28027891
·3732	·26727164	·3777	·27162978	·3822	·27599813	·3867	·28037630
·3733	·26736838	·3778	·27172675	·3823	·27609532	·3868	·28047370
·3734	·26746511	·3779	·27182372	·3824	·27619251	·3869	·28057111
·3735	·26756186	·3780	·27192069	·3825	·27628971	·3870	·28066852
·3736	·26765861	·3781	·27201767	·3826	·27638691	·3871	·28076594
·3737	·26775536	·3782	·27211466	·3827	·27648412	·3872	·28086336
·3738	·26785212	·3783	·27221165	·3828	·27658133	·3873	·28096078
·3739	·26794889	·3784	·27230864	·3829	·27667855	·3874	·28105821
·3740	·26804566	·3785	·27240565	·3830	·27677577	·3875	·28115564
·3741	·26814243	·3786	·27250265	·3831	·27687300	·3876	·28125308
·3742	·26823921	·3787	·27259966	·3832	·27697023	·3877	·28135052
·3743	·26833601	·3788	·27269668	·3833	·27706746	·3878	·28144797
·3744	·26843280	·3789	·27279370	·3834	·27716470	·3879	·28154542
·3745	·26852960	·3790	·27289072	·3835	·27726195	·3880	·28164288
·3746	·26862640	·3791	·27298775	·3836	·27735920	·3881	·28174034
·3747	·26872320	·3792	·27308479	·3837	·27745645	·3882	·28183781
·3748	·26882001	·3793	·27318183	·3838	·27755371	·3883	·28193528
·3749	·26891683	·3794	·27327887	·3839	·27765098	·3884	·28203275
·3750	·26901365	·3795	·27337592	·3840	·27774825	·3885	·28213023
·3751	·26911048	·3796	·27347298	·3841	·27784352	·3886	·28222772
·3752	·26920731	·3797	·27357004	·3842	·27794280	·3887	·28232520
·3753	·26930415	·3798	·27366710	·3843	·27804008	·3888	·28242270
·3754	·26940099	·3799	·27376417	·3844	·27813737	·3889	·28252019
·3755	·26949784	·3800	·27386125	·3845	·27823466	·3890	·28261770
·3756	·26959469	·3801	·27395833	·3846	·27833196	·3891	·28271520
·3757	·26969155	·3802	·27405541	·3847	·27842926	·3892	·28281272

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·3893	·28291023	·3938	·28730309	·3983	·29170475	·4028	·29611482
·3894	·28300775	·3939	·28740081	·3984	·29180266	·4029	·29621291
·3895	·28310528	·3940	·28749853	·3985	·29190057	·4030	·29631101
·3896	·28320281	·3941	·28759626	·3986	·29199849	·4031	·29640911
·3897	·28330034	·3942	·28769400	·3987	·29209642	·4032	·29650722
·3898	·28339788	·3943	·28779173	·3988	·29219435	·3033	·29660533
·3899	·28349342	·3944	·28788948	·3989	·29229228	·4034	·29670344
·3900	·28359297	·3945	·28798722	·3990	·29239022	·4035	·29680156
·3901	·28369052	·3946	·28808497	·3991	·29248816	·4036	·29689968
·3902	·28378808	·3947	·28818273	·3992	·29258610	·4037	·29699781
·3903	·28388564	·3948	·28828049	·3993	·29268405	·4038	·29709594
·3904	·28398321	·3949	·28837825	·3994	·29278200	·4039	·29719407
·3905	·28408078	·3950	·28847602	·3995	·29287996	·4040	·29729221
·3906	·28417835	·3951	·28857379	·3996	·29297792	·4041	·29739035
·3907	·28427593	·3952	·28867157	·3997	·29307589	·4042	·29748850
·3908	·28437351	·3953	·28876935	·3998	·29317386	·4043	·29758665
·3909	·28447110	·3954	·28886713	·3999	·29327183	·4044	·29768480
·3910	·28456870	·3955	·28896492	·4000	·29336981	·4045	·29778296
·3911	·28466629	·3956	·28906272	·4001	·29346779	·4046	·29788112
·3912	·28476389	·3957	·28916051	·4002	·29356577	·4047	·29797928
·3913	·28486150	·3958	·28925832	·4003	·29366376	·4048	·29807745
·3914	·28495911	·3959	·28935612	·4004	·29376176	·4049	·29817562
·3915	·28505673	·3960	·28945393	·4005	·29385976	·4050	·29827380
·3916	·28515435	·3961	·28955175	·4006	·29395776	·4051	·29837198
·3917	·28525197	·3962	·28964957	·4007	·29405576	·4052	·29847017
·3918	·28534960	·3963	·28974739	·4008	·29415378	·4053	·29856835
·3919	·28544723	·3964	·28984522	·4009	·29425179	·4054	·29866655
·3920	·28554487	·3965	·28994305	·4010	·29434981	·4055	·29876474
·3921	·28564251	·3966	·29004089	·4011	·29444783	·4056	·29886294
·3922	·28574015	·3967	·29013873	·4012	·29454585	·4057	·29896114
·3923	·28583781	·3968	·29023657	·4013	·29464388	·4058	·29905935
·3924	·28593546	·3969	·29033442	·4014	·29474192	·4059	·29915756
·3925	·28603312	·3970	·29043228	·4015	·29483996	·4060	·29925578
·3926	·28613078	·3971	·29053013	·4016	·29493800	·4061	·29935400
·3927	·28622845	·3972	·29062799	·4017	·29503605	·4062	·29945222
·3928	·28632612	·3973	·29072586	·4018	·29513410	·4063	·29955045
·3929	·28642380	·3974	·29082373	·4019	·29523215	·4064	·29964868
·3930	·28652148	·3975	·29092160	·4020	·29533021	·4065	·29974691
·3931	·28661917	·3976	·29101948	·4021	·29542827	·4066	·29984515
·3932	·28671686	·3977	·29111736	·4022	·29552634	·4067	·29994339
·3933	·28681455	·3978	·29121525	·4023	·29562441	·4068	·30004164
·3934	·28691225	·3979	·29131314	·4024	·29572248	·4069	·30013988
·3935	·28700995	·3980	·29141104	·4025	·29582056	·4070	·30023814
·3936	·28710766	·3981	·29150894	·4026	·29591864	·4071	·30033639
·3937	·28720537	·3982	·29160684	·4027	·29601673	·4072	·30043466

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Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·4073	·30053292	·4118	·30495866	·4163	·30939166	·4208	·31383153
·4074	·30063119	·4119	·30505709	·4164	·30949025	·4209	·31393027
·4075	·30072946	·4120	·30515553	·4165	·30958884	·4210	·31402902
·4076	·30082774	·4121	·30525397	·4166	·30968744	·4211	·31412776
·4077	·30092602	·4122	·30535242	·4167	·30978604	·4212	·31422651
·4078	·30102430	·4123	·30545086	·4168	·30983465	·4213	·31432526
·4079	·30112259	·4124	·30554932	·4169	·30998325	·4214	·31442402
·4080	·30122088	·4125	·30564777	·4170	·31008186	·4215	·31452278
·4081	·30131917	·4126	·30574623	·4171	·31018048	·4216	·31462154
·4082	·30141747	·4127	·30584469	·4172	·31027910	·4217	·31472030
·4083	·30151577	·4128	·30594316	·4173	·31037772	·4218	·31481907
·4084	·30161408	·4129	·30604163	·4174	·31047634	·4219	·31491784
·4085	·30171239	·4130	·30614010	·4175	·31057497	·4220	·31501661
·4086	·30181070	·4131	·30623858	·4176	·31067360	·4221	·31511539
·4087	·30190902	·4132	·30633706	·4177	·31077223	·4222	·31521417
·4088	·30200734	·4133	·30643554	·4178	·31087087	·4223	·31531296
·4089	·30210566	·4134	·30653403	·4179	·31096951	·4224	·31541174
·4090	·30220399	·4135	·30663252	·4180	·31106816	·4225	·31551053
·4091	·30230232	·4136	·30673101	·4181	·31116681	·4226	·31560933
·4092	·30240066	·4137	·30682951	·4182	·31126546	·4227	·31570812
·4093	·30249899	·4138	·30692801	·4183	·31136411	·4228	·31580692
·4094	·30259734	·4139	·30702651	·4184	·31146277	·4229	·31590572
·4095	·30269568	·4140	·30712502	·4185	·31156143	·4230	·31600453
·4096	·30279403	·4141	·30722353	·4186	·31166009	·4231	·31610334
·4097	·30289239	·4142	·30732205	·4187	·31175876	·4232	·31620215
·4098	·30299075	·4143	·30742057	·4188	·31185743	·4233	·31630096
·4099	·30308911	·4144	·30751909	·4189	·31195611	·4234	·31639978
·4100	·30318747	·4145	·30761761	·4190	·31205478	·4235	·31649860
·4101	·30328584	·4146	·30771614	·4191	·31215346	·4236	·31659743
·4102	·30338421	·4147	·30781467	·4192	·31225215	·4237	·31669625
·4103	·30348259	·4148	·30791321	·4193	·31235084	·4238	·31679509
·4104	·30358097	·4149	·30801175	·4194	·31244953	·4239	·31689392
·4105	·30367935	·4150	·30811029	·4195	·31254822	·4240	·31699276
·4106	·30377774	·4151	·30820884	·4196	·31264692	·4241	·31709159
·4107	·30387613	·4152	·30830739	·4197	·31274562	·4242	·31719044
·4108	·30397452	·4153	·30840594	·4198	·31284482	·4243	·31728928
·4109	·30407292	·4154	·30850450	·4199	·31294303	·4244	·31738813
·4110	·30417132	·4155	·30860306	·4200	·31304174	·4245	·31748698
·4111	·30426973	·4156	30870162	·4201	·31314045	·4246	·31758584
·4112	·30436813	·4157	·30880019	·4202	·31323917	·4247	·31768470
·4113	·30446655	·4158	·30889876	·4203	·31333789	·4248	·31778356
·4114	·30456496	·4159	·30899733	·4204	·31343661	·4249	·31788242
·4115	·30466338	·4160	·30909591	·4205	·31353534	·4250	·31798129
·4116	·30476180	·4161	·30919449	·4206	·31363407	·4251	·31808016
·4117	·30486023	·4162	·30929307	·4207	·31373280	·4252	·31817903

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·4253	·31827791	·4298	·32273040	·4343	·32718864	·4388	·33165224
·4254	·31837679	·4299	·32282941	·4344	·32728777	·4389	·33175149
·4255	·31847567	·4300	·32292843	·4345	·32738691	·4390	·33185074
·4256	·31857455	·4301	·32302744	·4346	·32748605	·4391	·33195000
·4257	·31867344	·4302	·32312646	·4347	·32758529	·4392	·33204925
·4258	·31877233	·4303	·32322548	·4348	·32768434	·4393	·33214851
·4259	·31887123	·4304	·32332451	·4349	·32778348	·4394	·33224777
·4260	·31897013	·4305	·32342354	·4350	·32788263	·4395	·33234704
·4261	·31906903	·4306	·32352257	·4351	·32798179	4396	·33244630
·4262	·31916793	·4307	·32362160	·4352	·32808094	4397	·33254557
·4263	·31926684	·4308	·32372064	·4353	·32818010	4398	·33264484
·4264	·31936574	·4309	·32381968	·4354	·32827926	·4399	33274412
·4265	·31946466	·4310	·32391872	·4355	·32837842	·4400	·33284539
·4266	·31956357	·4311	·32401776	·4356	·32847759	·4401	·33294267
·4267	·31966249	·4312	·32411681	·4357	·32857670	·4402	·33304195
·4268	·31976141	·4313	·32421586	·4358	·32867593	·4403	33314124
·4269	·31986033	·4314	·32431491	·4359	32877510	·4404	·33324052
·4270	·31995926	·4315	·32441397	·4360	·32887428	·4405	·33333981
·4271	·32005819	·4316	·32451303	·4361	·32897346	·4406	·33343910
·4272	·32015712	·4317	·32461209	·4362	32907264	·4407	·33353840
·4273	·32025606	·4318	·32471115	·4363	·32917182	·4408	·33363769
·4274	·32035500	·4319	·32481022	·4364	·32927101	·4409	·33373699
·4275	·32045394	·4320	·32490929	·4365	·32937020	·4410	·33383629
·4276	·32055289	·4321	·32500836	·4366	·32946939	·4411	·33393559
·4277	·32065183	·4322	·32510744	·4367	·32956858	·4412	·33403490
·4278	·32075078	·4323	·32520651	·4368	·32966778	·4413	·33413420
·4279	·32084974	·4324	·32530559	·4369	·32976698	·4414	·33423351
·4280	·32094869	·4325	·32540468	·4370	·32986618	·4415	·33433282
·4281	·32104765	·4326	·32550376	4371	·32996538	·4416	·33443214
·4282	·32114661	4327	·32560285	·4372	·33006459	·4417	·33453146
·4283	·32124558	·4328	·32570194	·4373	·33016380	·4418	·33463078
·4284	·32134455	·4329	·32580104	·4374	·33026301	·4419	·33473010
·4285	·32144352	·4330	·32590013	·4375	33036223	·4420	·33482942
·4286	·32154249	·4331	·32599923	·4376	·33046144	·4421	·33492875
·4287	·32164147	·4332	·32609833	·4377	·33056066	·4422	·33502807
·4288	·32174045	·4333	·32619744	·4378	·33065988	·4423	·33512741
·4289	·32183943	·4334	·32629655	·4379	33075911	·4424	·33522674
·4290	·32193842	·4335	·32639566	4380	·33085834	·4425	·33532607
·4291	·32203740	·4336	32649477	·4381	·33095757	·4426	·33542541
·4292	·32213640	·4337	·32659389	·4382	33105680	·4427	·33552475
·4293	·32223539	·4338	·32669301	4383	33115603	·4428	·33562409
·4294	·32233439	·4339	·32679213	·4384	·33125527	·4429	·33572344
·4295	·32243339	·4340	·32689125	·4385	·33135451	·4430	·33582279
·4296	·32253239	·4341	·32699038	·4386	·33145375	·4431	33592213
·4297	·32263139	·4342	·32708951	·4387	·33155299	·4432	·33602149

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·4433	·33612084	·4478	·34059400	·4523	·34507154	·4568	·34955289
·4434	·33622020	·4479	·34069352	·4524	·34517108	·4569	·34965252
·4435	·33631955	·4480	·34079297	·4525	·34527063	·4570	·34975215
·4436	·33641892	·4481	·34089243	·4526	·34537018	·4571	·34985178
·4437	·33651828	·4482	·34099189	·4527	·34546973	·4572	·34995141
·4438	·33661764	·4483	·34109136	·4528	·34556928	·4573	·35005104
·4439	·33671701	·4484	·34119082	·4529	·34566883	·4574	·35015068
·4440	·33681638	·4485	·34129029	·4530	·34576839	·4575	·35025031
·4441	·33691575	·4486	·34138976	·4531	·34586795	·4576	·35034995
·4442	·33701513	·4487	·34148923	·4532	·34596751	·4577	·35044959
·4443	·33711450	·4488	·34158870	·4533	·34606707	·4578	·35054924
·4444	·33721388	·4489	·34168818	·4534	·34616663	·4579	·35064888
·4445	·33731326	·4490	·34178765	·4535	·34626620	·4580	·35074853
·4446	·33741265	·4491	·34188713	·4536	·34636577	·4581	·35084817
·4447	·33751203	·4492	·34198662	·4537	·34646534	·4582	·35094782
·4448	·33761142	·4493	·34208610	·4538	·34656491	·4583	·35104747
·4449	·33771081	·4494	·34218558	·4539	·34666448	·4584	·35114713
·4450	33781020	·4495	·34228507	·4540	·34676406	·4585	·35124678
·4451	·33790959	·4496	·34238456	·4541	·34686363	·4586	·35134644
·4452	·33800899	·4497	·34248405	·4542	·34696321	·4587	·35144609
·4453	·33810839	·4498	·34258355	·4543	·34706279	·4588	·35154575
·4454	·33820779	·4499	·34268304	·4544	·34716237	·4589	·35164541
·4455	·33830719	·4500	·34278254	·4545	·34726196	·4590	·35174508
·4456	·33840660	·4501	·34288204	·4546	·34736154	·4591	·35184474
·4457	·33850601	·4502	·34298154	·4547	·34746113	·4592	·35194441
·4458	·33860542	·4503	·34308105	·4548	·34756072	·4593	·35204407
·4459	·33870483	·4504	·34318055	·4549	·34766031	·4594	·35214374
·4460	·33880424	·4505	·34328006	·4550	·34775991	·4595	·35224341
·4461	·33890366	·4506	·34337957	·4551	·34785950	·4596	·35234308
·4462	·33900308	·4507	·34347908	·4552	·34795910	·4597	·35244276
·4463	·33910250	·4508	·34357859	·4553	·34805870	·4598	·35254243
·4464	·33920192	·4509	·34367811	·4554	·34815830	4599	·35264211
·4465	·33930134	·5510	·34377763	·4555	·34825790	·4600	·35274179
·4466	·33940077	·4511	·34387715	·4556	·34835750	·4601	·35284147
·4467	·33950020	·4512	·34397667	·4557	·34845711	·4602	·35294115
·4468	·33959963	·4513	·34407619	·4558	·34855672	·4603	·35304084
·4469	·33969907	·4514	·34417572	·4559	·34865633	·4604	·35314052
·4470	·33979850	·4515	·34427525	·4560	·34875594	·4605	·35324021
·4471	·33989794	·4516	·34437478	·4561	·34885555	·4606	·35333990
·4472	·33999738	·4517	·34447431	·4562	·34895516	·4607	·35343958
·4473	·34009682	·4518	·34457384	·4563	·34905478	·4608	·35353928
·4474	·34019626	·4519	·34467338	·4564	·34915440	·4609	·35363897
·4475	·34029571	·4520	·34477291	·4565	·34925402	·4610	·35373866
·4476	·34039516	·4521	·34487245	·4566	·34935364	·4611	·35383836
·4477	·34049461	·4522	·34497199	·4567	·34945326	·4612	·35393806

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·4613	·35403776	·4658	·35852577	·4703	·30301656	·4748	·36750976
·4614	·35413746	·4659	·35862553	·4704	·36311638	·4749	·36760963
·4615	·35423716	·4660	·35872530	·4705	·36321621	·4750	·36770950
·4616	·35433686	·4661	·35882507	·4706	·36331603	·4751	·36780938
·4617	·35443657	·4662	·35892484	·4707	·36341586	·4752	·36790925
·4618	·35453628	·4663	·35902461	·4708	·36351569	·4753	·36800913
·4619	·35463598	·4664	·35912439	·4709	·36361552	·4754	·36810901
·4620	·35473569	·4665	·35922416	·4710	·36371535	·4755	·36820889
·4621	·35483541	·4666	·35932394	·4711	·36381518	·4756	·36830877
·4622	·35493512	·4667	·35942372	·4712	·36391501	·4757	·36840865
·4623	·35503483	·4668	·35952349	·4713	·36401485	·4758	·36850853
·4624	·35513455	·4669	·35962327	·4714	·36411469	·4759	·36860842
·4625	·35523427	·4670	·35972306	·4715	·36421452	·4760	·36870830
·4626	·35533399	·4671	·35982284	·4716	·36431436	·4761	·36880819
·4627	·35543371	·4672	·35992262	·4717	·36441420	·4762	·36890807
·4628	·35553343	·4673	·36002241	·4718	·36451404	·4763	·36900796
·4629	·35563315	·4674	·36012219	·4719	·36461388	·4764	·36910785
·4630	·35573288	·4675	·36022198	·4720	·36471372	·4765	·36920774
·4631	·35583260	·4676	·36032177	·4721	·36481357	·4766	·36930763
·4632	·35593233	·4677	·36042156	·4722	·36491341	·4767	·36940752
·4633	·35603206	·4678	·36052135	·4723	·36501326	·4768	·36950741
·4634	·35613179	·4679	·36062115	·4724	·36511310	·4769	·36960730
·4635	·35623153	·4680	·36072094	·4725	·36521295	·4770	·36970720
·4636	·35633126	·4681	·36082074	·4726	·36531280	·4771	·36980709
·4637	·35643099	·4682	·36092053	·4727	·36541265	·4772	·36990699
·4638	·35653073	·4683	·36102033	·4728	·36551250	·4773	·37000688
·4639	·35663047	·4684	·36112013	·4729	·36561236	·4774	·37010678
·4640	·35673021	·4685	·36121993	·4730	·36571221	·4775	·37020668
·4641	·35682995	·4686	·36131973	·4731	·36581206	·4776	·37030658
·4642	·35692969	·4687	·36141954	·4732	·36591192	·4777	·37040648
·4643	·35702944	·4688	·36151934	·4733	·36601178	·4778	·37050638
·4644	·35712918	·4689	·36161915	·4734	·36611163	·4779	·37060628
·4645	·35722893	·4690	·36171895	·4735	·36621149	·4780	·37070618
·4646	·35732868	·4691	·36181876	·4736	·36631135	·4781	·37080609
·4647	·35742843	·4692	·36191857	·4737	·36641121	·4782	·37090599
·4648	·35752818	·4693	·36201838	·4738	·36651108	·4783	·37100590
·4649	·35762793	·4694	·36211820	·4739	·36661094	·4784	·37110580
·4650	·35772769	·4695	·36221801	·4740	·36671080	·4785	·37120571
·4651	·35782744	·4696	·36231783	·4741	·36681067	·4786	·37130562
·4652	·35792720	·4697	·36241764	·4742	·36691053	·4787	·37140553
·4653	·35802696	·4698	·36251746	·4743	·36701040	·4788	·37150544
·4654	·35812672	·4699	·36261728	·4744	·36711027	·4789	·37160535
·4655	·35822648	·4700	·36271710	·4745	·36721014	·4790	·37170526
·4656	·35832624	·4701	·36281691	·4746	·36731001	·4791	·37180517
·4657	·35842600	·4702	·36291673	·4747	·36740988	·4792	·37190508

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Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·4793	·37200500	·4838	·37650192	·4883	·38100015	·4928	·38549933
·4794	·37210491	·4839	·37660186	·4884	·38110012	·4929	·38559932
·4795	·37220483	·4840	·37670181	·4885	·38120010	·4930	·38569931
·4796	·37230474	·4841	·37680176	·4886	·38130007	·4931	·38579930
·4797	·37240466	·4842	·37690171	·4887	·38140004	·4932	·38589929
·4798	·37250458	·4843	·37700166	·4888	·38150002	·4933	·38599928
·4799	·37260450	·4844	·37710161	·4889	·38159999	·4934	·38609927
·4800	·37270442	·4845	·37720156	·4890	·38169997	·4935	·38619926
·4801	·37280434	·4846	·37730152	·4891	·38179994	·4936	·38629926
·4802	·37290426	·4847	·37740147	·4892	·38189992	·4937	·38639925
·4803	·37300418	·4848	·37750142	·4893	·38199990	·4938	·38649924
·4804	·37310410	·4849	·37760138	·4894	·38209988	·4939	·38659923
·4805	·37320403	·4850	·37770133	·4895	·38219985	·4940	·38669923
·4806	·37330395	·4851	·37780129	·4896	·38229983	·4941	·38679922
·4807	·37340387	·4852	·37790124	·4897	·38239981	·4942	·38689921
·4808	·37350380	·4853	·37800120	·4898	·38249979	·4943	·38699920
·4809	·37360373	·4854	·37810116	·4899	·38259977	·4944	·38709920
·4810	·37370365	·4855	·37820111	·4900	·38269975	·4945	·38719919
·4811	·37380358	·4856	·37830107	·4901	·38279973	·4946	·38729919
·4812	·37390351	·4857	·37840103	·4902	·38289971	·4947	·38739918
·4813	·37400344	·4858	·37850099	·4903	·38299969	·4948	·38749918
·4814	·37410337	·4859	·37860095	·4904	·38309967	·4949	·38759917
·4815	·37420330	·4860	·37870091	·4905	·38319965	·4950	·38769917
·4816	·37430324	·4861	·37880087	·4906	·38329963	·4951	·38779916
·4817	·37440317	·4862	·37890083	·4907	·38339961	·4952	·38789916
·4818	·37450310	·4863	·37900080	·4908	·38349960	·4953	·38799915
·4819	·37460304	·4864	·37910076	·4909	·38359958	·4954	·38809915
·4820	·37470297	·4865	·37920072	·4910	·38369956	·4955	·38819914
·4821	·37480291	·4866	·37930069	·4911	·38379955	·4956	·38829914
·4822	·37490284	·4867	·37940065	·4912	·38389953	·4957	·38839913
·4823	·37500278	·4868	·37950062	·4913	·38399952	·4958	·38849913
·4824	·37510272	·4869	·37960058	·4914	·38409950	·4959	·38859913
·4825	·37520266	·4870	·37970055	·4915	·38419949	·4960	·38869912
·4826	·37530259	·4871	·37980051	·4916	·38429947	·4961	·38879912
·4827	·37540253	·4872	·37990048	·4917	·38439946	·4962	·38889912
·4828	·37550247	·4873	·38000045	·4918	·38449945	·4963	·38899912
·4829	·37560242	·4874	·38010042	·4919	·38459943	·4964	·38909911
·4830	·37570236	·4875	·38020038	·4920	·38469942	·4965	·38919911
·4831	·37580230	·4876	·38030035	·4921	·38479941	·4966	·38929911
·4832	·37590224	·4877	·38040032	·4922	·38489939	·4967	·38939911
·4833	·37600219	·4878	·38050029	·4923	·38499938	·4968	·38949910
·4834	·37610213	·4879	·38060026	·4924	·38509937	·4969	·38959910
·4835	·37620208	·4880	·38070023	·4925	·38519936	·4970	·38969910
·4836	·37630202	·4881	·38080020	·4926	·38529935	·4971	·38979910
·4837	·37640197	·4882	·38090018	·4927	·38539934	·4972	·38989910

Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area	Verſ. ſine	Seg. area
·4973	·38999909	·4980	·39069909	·4987	·39139908	·4994	·39209908
·4974	·39009909	·4981	·39079909	·4988	·39149908	·4995	·39219908
·4975	·39019909	·4982	·39089908	·4989	·39159908	·4996	·39229908
·4976	·39029909	·4983	·39099908	·4990	·39169908	·4997	·39239908
·4977	·39039909	·4984	·39109908	·4991	·39179908	·4998	·39249908
·4978	·39049909	·4985	·39119908	·4992	·39189908	·4999	·39259908
·4979	·39059909	·4986	·39129908	·4993	·39199908	·5000	·39269908

In the preceding table, each number in the column of ſegments is the area of the circular ſegment whoſe height or verſed ſine of its half arc is the number immediately on the left of it, the diameter of the circle being 1, and the whole area ·78539816. This table is here extended to ten times the length which it had uſually been, the diameter being divided into ten thouſand equal parts inſtead of one thouſand: And this ſize of it alſo allowed me to carry each number out to eight places of figures inſtead of ſix; for when the diameter is divided into only one thouſand parts, the proportional part for the figures in the verſed ſine beyond the third will not give the area true beyond the 6th figure; but in this table it is always found true in the 8th place.

Rule the 8th to the circular ſegment in page 108 exto add there, when a ſegment greater than a ſemicircle is plains the uſe of it; to which it may only be neceſſary to be found, ſubtract the quotient of its verſed ſine divided by its diameter from 1, then ſubtract the tabular ſegment which correſponds to the remainder from ·78539816 the whole tabular circle, and the remainder will be the tabular ſegment correſponding to the ſegment required, and which muſt then be multiplied by the ſquare of the diameter.

F I N I S.

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E R R A T A.

Page	*Line*	*For*	*Read*
1	7	ſtroke	trace of a pen &c.
17	laſt	$DF : EG$	$EG : DF$
28	laſt	prob. 7	prob. 6
30	5*b*	$\frac{aa - cc}{b} +$	$\frac{aa - cc}{b} =$
31	13	c^5	$3c^5$
32	7*b*	$\frac{aa}{aa}$	$\frac{aa}{rr}$
36	10	eye	center of the quadrant
37	1	2·7473774	2·7474774
	2	219·790192	219·798192
39	3	2·0845048	2·0845333, which will make the laſt line of that ex. become $62·623 = AD$
41	4*b*	$\angle DCB$	$\angle DBC$
4	7	8·04	8·40
47	5	53° 20′	53° 30′, which being corrected throughout the example, will make the laſt line of it become $HM = 479·7933$
51	12	8·6513363	8·6513357
61	12*b*	245 Acres	24·5 Acres
67	3	61 ac. 3 r. 36·8064 p.	61 ac. 1 r. 39·68 p.
70	12	greater ſum from the leſs	leſs ſum from the greater
74	9*b*	$\frac{AB + CD}{2}$	$\frac{AB + CD}{2} \times AP$
75		*In the 2d ex. ſay* ſee fig. in p. 69	
98	10 16	radius	diameter
103	3*b*	$n = \frac{1}{1.1.3} + \&c.$	$n = 2 \times : \frac{1}{1.1.3} + \&c.$
112	laſt	he	the
115		In the triangle *T*, ſupply the *T*	
120	3	ar	lar
	ib.	$\sqrt{BC - CD^2}$	$\sqrt{BC^2 - CD^2}$
126	17	16·944305	16·944302
133	11*b*	$KE =$	$KE = 13$
138	7	extentions	extenſions
140	6	rowling	rolling
144	12	feet to the lb.	lb. to the foot
154	14	100	110
160	6	5 × 15	15 × 15
164	2	diſſimular	diſſimilar
ib.	7*b*	*at the end add the word* and	
174	9	corollary 2	corollary 1
175	13	$20 - 30 - \overline{19\frac{1}{5}}$	$20 - \overline{30 - 19\frac{1}{5}}$

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ERRATA.

Page	*Line*	*For*	*Read*
175	5 *b*	*CP* being	(*CP* being
178	10 *b*	1606c·41934	1606·41834
200	8	wil	will
201	9	263857624944	263858149120
217	11	parabolid	paraboloid
236	7	$x\sqrt{\frac{aa-\frac{1}{3}dxx}{aa-\frac{1}{3}xx}}\times$	$x\sqrt{\frac{aa-\frac{1}{3}dxx}{aa-\frac{1}{3}xx}}$ is $= x \times$
405	11	ſolidity	ſuperficies
414	7 *b*	$18+2=20=$ diameter	$18+4=22=$ diameter, which will make the whole expence become 3 *l.* 12 *s.* $4\frac{1}{2}$ *d.* This will alſo alter the contents in the example in the following page, the number of pounds being 240 and the price 4 *l.*
428	5	128×6^1	128×6
443	11 *b*	$b-x$	$b+x$
ib.	6 *b*	$\frac{5\cdot7\cdot9}{4.6.8}-$	$\frac{5\cdot7\cdot9}{4.6.8}+$
	ib.	$\frac{1+\sqrt{2}}{3\sqrt{2}}=$	$\frac{1+\sqrt{2}}{3\sqrt{2}}=\frac{2+\sqrt{2}}{6}=$
452	14	duadrable	quadrable
459	9	or (*EG*)	(or *EG*)
469	8	25 *C*	50 *C*
ib.	14	ſecond	third
489	3	nothings	northings
ib.	*ib.*	*ib.*	
492		At the bottom of the fig. draw a line through all the points *a i b c h e d g f.*	
496	21	78	102
ib.	4 *b*	*ib.*	
524	9 *b*	paraboidal	paraboloidal
526	1	*After the words* then if the ſum be, *add the words* multiplied by the length, and the product be	
531	1 *b* 2 *b*	*dele the words* excepting the allowance, &c.	
539	22	Briton	Britain
549	8	Square-roots	ſquares
600	15	thick	broad

N. B. *b* at the line figures denotes them to be counted from the bottom.

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